THE FORTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The forty-fifth regular meeting of the San Francisco Section of this Society was held at Stanford University on April 4, 1925. Professor E. R. Hedrick presided at the morning session and Professor A. F. Carpenter at the afternoon session. The total attendance was thirty-six, including the following twenty-six members of the Society:

Allardice, A. D. B. Andrews, Bernstein, Blichfeldt, Buck, Cajori, A. F. Carpenter, Daus, M. W. Haskell, E. R. Hedrick, Hoskins, Hotelling, Glenn James, Lehmer, Sophia Levy, E. W. Mc Donald, J. H. Mc Donald, W. A. Manning, Mathewson, Moreno, F. R. Morris, Noble, Pauline Sperry, Stager, A. R. Williams, Wong.

It was voted that, beginning with the next Fall meeting, the retiring Chairman deliver an address before the Section.

The dates and places of the next three regular meetings of the Section were fixed as follows: June 19, 1925, University of Oregon; October 31, 1925, University of California; April 3, 1926, Stanford University.

Titles and abstracts of papers read at this meeting follow. The papers of Dr. Barter, Dr. Mullemeister, and of Professors Bateman, Bell, Moritz were read by title.

1. Dr. John Barter: *Notes on algebraic theory. Part* I. *The linear operator.*

The present paper developes the theory of linear transformations in such directions as seem essential to application to the general algebraic theory. The theory of λ-matrices is treated with some detail from a new angle and by a method applicable to λ-matrices of any degree. The new conception of poles of different orders is made a basis for a classification of these matrices. The classic theory of elementary divisors appears as a particular case of the general theory. Automorphic transformations are treated incidentally and in an introductory manner from this standpoint.

2. Professor Harry Bateman: *An extension of Lagrange's expansion.*

When z is defined by means of the equation $z = a + x\Phi(z)$, Lagrange's theorem furnishes the expansion, in ascending powers of x, of an arbitrary function $f(z)$. The present extension of the expansion gives a corresponding series for the definite integral

$$\int_a^z F[a + x\Phi(\tau) - \tau, \tau]d\tau$$

where F is an arbitrary function of its two arguments. The series was suggested by a generalization of E. T. Whittaker's solution of Laplace's equation.

3. Professor E. T. Bell: *The chi functions of Glaisher and class numbers.*

The function $\chi_r(n)$ of Glaisher is the sum of the rth powers of all Gaussian primary complex numbers having norm n. This function is here expressed in terms of binary quadratic class numbers, and several new class number relations are indicated.

4. Professor B. A. Bernstein: *On the existence of fields in boolean algebras.*

The author proves that $xy' + x'y$, xy; $xy + x'y'$, $x + y$ (x' the negative of x) are the only pairs of operations for which the elements of a two-element boolean algebra are a field, and that there exist no field operations expressible in terms of addition, multiplication, and negation for the totality of the elements of any other boolean algebra.

5. Professor H. F. Blichfeldt: *On the minimum value of positive real quadratic forms in 6 variables.*

Let f represent a positive real quadratic form in n variables, having a determinant D. Then we may substitute integers, not all zero, for the variables such that the numerical value of f is not greater than $\gamma_n \cdot \sqrt[n]{|D|}$; where γ_n is a function of n only. Superior limits to γ_n have been given for any n, as well as the exact minima for $n = 2, 3, 4, 5$ (cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), p. 233). In the present paper the author shows that the exact minimum for $n = 6$ is $2/\sqrt[6]{3}$.

6. Professor Florian Cajori: *American contributions to mathematical notations.*

The author makes reference to the Maya numeral notation marking the earliest regular use of a sign for zero and the principle of local value, the Peruvian knot records, the dollar mark, a sporadic notation for radicals, the sign $\overset{0}{=}$ for equivalence, B. Peirce's symbols for π, e and i, C. S. Peirce's notation in symbolic logic, the $\doteq$ for approach to the limit, J. W. Gibbs's vector notation, E. H. Moore's notation in general analysis. As yet, none of these have found permanent adoption in Europe, except perhaps the dollar mark.

7. Professor Florian Cajori: *Early appearance in print of the dollar mark.*

The author shows that the dollar mark originated from the superposition of the letters p and s in "pesos". The frequent statement that the first printed appearance of the sign \$ to represent dollars is in Chauncey Lee's *American Accomptant*, 1797, was shown to be inaccurate. The \$ occurred in print in the *Scholars' Arithmetic* of Daniel Adams (4th. ed., 1807) and began to be used freely in newspapers about 1810.

8. Professor A. F. Carpenter: *Two point-line correspondences determined by a ruled surface.*

By means of the two quadrics which osculate the two sheets of the flecnode surface of a ruled surface along a pair of corresponding line elements, two point-line correspondences of space are set up. These correspondences are such that all points of space are made to correspond, in two ways, to the lines of a congruence.

The paper also determines for numerous point loci of space, the corresponding line loci.

9. Professor P. H. Daus: *Certain recursion formulas connected with the solution of* $x^2+y^2=z^2$.

If

$$s_1 = (x_1, y_1, z_1)$$

is the smallest primitive solution of

$$x^2+y^2=z^2,$$

where

$$y = x + m,$$

then all solutions of these equations when m is a prime or the power of a prime are given by the formulas

$$s_{2n+1} = ps_{2n} - ps_{2n-1} + s_{2n-2};$$
$$s_{2n+2} = qs_{2n+1} - qs_{2n} + s_{2n-1};$$

where p and q depend on s_1. The solutions divide themselves into two sets, according as n of s_n is even or odd. Within each set the solutions are connected by the relation

$$s_{n+2} = 7s_{n+1} - 7s_n + s_{n-1},$$

independent of the value of m.

10. Professor M. W. Haskell: *Autopolar configurations in space.*

The method of finding multiply self-polar curves presented to the International Congress at Toronto can be extended to configurations in space. There are certain modifications to be noted, in particular with regard to the nature of the combination of a polarity with a collineation group of which the quadric surface in question is an invariant. Perspective reflexions always give rise to new polarities, while skew reflexions are of two kinds. If the invariant lines of a skew reflexion are polar to each other, then the product of the collineation and the given polarity is a second polarity; while if they are self-polar (i. e. tangent to the given quadric surface) the resulting correlation is a null-system.

For example, the Kummer surface is invariant under a group of 16 collineations of period 2 (all *skew* reflexions) and 16 correlations, of which 10 are polarities with respect to quadric surfaces and 6 are polarities with respect to null-systems.

11. Professor E. R. Hedrick: *Extensions of Picard's theorem to non-analytic functions.*

The author points out that there are a number of rather general cases in which extensions of Picard's theorem to non-analytic functions are possible, and several specific forms of the extended theorem are given. Some limitations on possible extensions are stated, for functions that are non-analytic.

12. Dr. Harold Hotelling: *Theory of mine economics.*

The equations of competition are modified in the case of exhaustible resources by the addition to each equation of a term of the form $\lambda_i e^{\delta t}$, where δ is the force of interest and the λ_i's depend upon the amounts of the resource in the hands of the various entrepreneurs. Although the the equations of competition of inexhaustible resources admit of solution when demand is assumed perfectly inelastic, this assumption in the present case leads to an absurdity. Though natural resources may in some cases be exploited too rapidly under competition, the analysis shows that in other cases the exploitation is slower than the public interest requires.

13. Dr. Harold Hotelling: *Manifolds of states of motion as multiple-sheeted spaces.*

The boundaries of a pair of plane rings which link form the branch curves of a $(2p+2)$-sheeted space homeomorphic with the manifold of states of motion on an orientable surface of genus $p>0$. For a one-sided surface of the same connectivity the representation is by branch curves of the same type in a space of $2p+4$ sheets. For the representation of the states of motion on a one-sided surface of the first kind the branch system consists of these four curves and two others, each of the latter linking doubly the other and also one of the plane rings.

14. Professor Glenn James: *On the solution of equations of degree higher than the fifth.*

This paper developes a formula for one root of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0,$$

when n is odd or when n is even and a_n is negative, and presents a theoretically possible method of solving this equation for any positive integral value of n.

15. Professor D. N. Lehmer: *A new method of finding factors of numbers.*

The author's method is an improvement on the well known method which depends on the theory of quadratic residues. By means of a set of stencils the divisors

which belong to a set of linear forms are made apparent at once. The method is available for numbers as high as ten billions.

16. Professor J. H. McDonald: *The equation of the elliptic cylinder.*

The solution of the linear differential equation of the second order which belongs to the elliptic cylinder is given for the general case. The equation is of a type not derivable by specialization from the hypergeometric differential equation and has a non-regular singular point at infinity. The problem is to determine the characteristic exponent and the coefficients of the series giving the solution, the coefficients being functions of the characteristic exponent.

The differential equation depends on two parameters and is first transformed into a Riccati equation which necessarily has a periodic solution. This is obtained in the form of a series of powers of one of the parameters, the coefficients being periodic functions of the independent variable. The sum of the constant terms in these coefficients is a power series in one parameter, the coefficients being rational functions of the other. This series is a branch of the characteristic exponent. The coefficients of the solution are next given explicitly by means of infinite determinants. The equation for the characteristic exponent which arises in determining the coefficients is put in a form such that all branches of the exponent can be determined.

17. Professor J. H. McDonald: *The periodic solutions of the equation of the elliptic cylinder.*

The solution of the equation of the elliptic cylinder is not periodic in general but only if a special relation holds between the parameters. The forms of a few of the periodic solutions have been determined but by a method which does not seem capable of being extended to the general case. In this paper a method is given by which they may all be obtained, and the equation between the parameters which must be satisfied for the existence of the periodic solution is completely solved. The periodic solutions are known to be the solutions of a particular integral equation. Its general solution and the characteristic numbers are accordingly determined.

18. Professor J. H. McDonald: *On the periodic solution of a certain differential equation.*

A problem in motion under a periodic disturbing force leads to the differential equation

$$\frac{d^2x}{dt^2} - mx\left(\frac{dx}{dt}\right)^2 + n^2x = a \sin pt.$$

This equation has a single periodic solution if n^2/p^2 is not equal to the square of an odd integer. The solution is of the form

$$\sum_1^\infty a_k \sin (2k-1)\, pt$$

and the coefficients a_k are all determined. The method used is that of solving a succession of differential equations which have the given equation as a limiting form; the solution at each step being chosen so that a succession of functions is determined which converges to a solution of the problem.

19. Professor J. H. McDonald: *The transformation of elliptic integrals.*

The theory of transformation of elliptic integrals given previously by the writer for the general case is applied to the transformation of the seventh order. The modular equation is obtained in the appropriate form and the results compared with those of Kiepert and Joubert.

20. Professor J. H. McDonald: *On the reduction of the singularities of algebraic curves.*

Halphen enunciated the theorem that an algebraic curve can be transformed birationally into one whose multiple points are double points with distinct tangents. His proof was defective but proofs have been given by Poincaré, Bertini, and others. A short proof of the theorem is given by determining a space curve whose projection is the required curve with reduced singularities.

21. Professor R. E. Moritz: *On the kinematic construction of higher plane curves.*

Professor Moritz considers the various loci which result from a composition of certain elementary motions which

can readily be effected kinematically. The equations of these motions give in each case the simplest possible parametric equations of the loci under consideration. This paper will appear in the AMERICAN MATHEMATICAL MONTHLY.

22. Dr. Hermance Mullemeister: *A projection of the lines in space into pairs of lines in a plane.*

The four-dimensional line-space is projected from two centers C_1 and C_2 into pairs of lines in a plane. An axial complex on C_1 C_2 is projected into a pencil of lines on C_0, the intersection of C_1 C_2 and the picture plane α. A flat pencil is projected into two perspective pencils in α, whose vertices P_1 and P_2 are collinear with C_0 and whose axis of perspectivity is the intersection of the two planes. Two pencils P_1 and P_2 in α which are projective, but not perspective, constitute the image of a quadratic ruled system, whose axes are P_1C_1 and P_2C_2. A quadratic ruled system which does not pass through C_1 and C_2, is projected into two projective line-conics which have two common tangents through C_0.

A line-field in a plane β is projected into two perspective line-fields in α having C_0 as center of perspectivity and the intersection of α and β as axis of perspectivity. A line-sheaf on O is projected into two pencils whose vertices are the projections of O from C_1 and C_2, which are collinear with C_0. A bilinear congruence is projected into pairs of lines such that to every line-pencil (l_1) corresponds a line-conic μ_2^2, and to every line-pencil (l_2) corresponds a line-conic μ_1^2.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING IN CHICAGO

The twenty-third Western meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10 and 11, 1925.

Ninety-six persons registered at this meeting, among whom were the following eighty-one members of the Society:

F. E. Allen, R. W. Babcock, Barnard, W. S. Beckwith, H. A. Bender, Birkhoff, Bubb, C. C. Camp, Carmichael, Carr, Chittenden, Crathorne, Curtiss, Dickson, Dostal, Dresden, Emch, Feltges, C. A. Garabedian, Garver, Gorrell, V. G. Grove, W. L. Hart, Hildebrandt, Holl, L. A. Hopkins, Mildred Hunt, Huntington, Louis Ingold, Ingraham, Dunham Jackson, Kempner, Kerékjártó, Kinney, Kouperman, Krathwohl, E. P. Lane, Laves, Libman, Logsdon, MacDuffee, W. D. MacMillan, March, William Marshall, T. E. Mason, Mathews, B. I. Miller, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, F. S. Nowlan, Parkinson, Reinsch, H. L. Rietz, Roever, Roth, Rowland, Runge, Schottenfels, Showman, Simmons, W. G. Simon, E. B. Skinner, Slaught, E. R. Smith, H. L. Smith, Steimley, Stouffer, Teach, E. L. Thompson, Townsend. J. S. Turner, Van Vleck, Wahlin, E. M. Weaver, K. P. Williams, Frederick Wood, F. E. Wood, J. M. Young, J. W. A. Young.

At the Council meeting the following elections to membership took place:

To sustaining membership:

Amherst College, Amherst, Mass.;
Harvard University, Cambridge, Mass.;
The Maccabees, Detroit, Mich.;
The Missouri State Life Insurance Company, St. Louis, Mo.;
The Pacific Mutual Life Insurance Company, Los Angeles, Cal.;
The Prudential Insurance Company, Newark, N. J.;
The Travelers Insurance Company, Hartford, Conn.

To ordinary membership:

Professor Ethel Beatrice Callahan, Cedar Crest College;
Mr. Claude Bernhart Dansby, Morehouse College;
Mr. Harold John Hornberger, Great Northern Life Insurance Company;
Professor Béla de Kerékjártó, University of Szeged;
Mr. Donald D. Laun, University of Chicago;
Rev. Paul Muehlman, Loyola University;
President Ellen Fitz Pendleton, Wellesley College;
Mr. Clarence Edward Rose, Arkansas Cold Storage Company;
Dr. Charles Edward St. John, Mount Wilson Observatory;
Professor Jennie Leatitia Tate, McMurry College;
Rev. George Warren Walker, Margaretville, N. Y.

Nominees of the John Hancock Life Insurance Company, Boston, Mass.:

Eleanor Alice Abbott; Harold A. Garabedian; Harold Alden Grant; L. H. Howe; Earl M. Thomas.

Nominees of the Connecticut Mutual Life Insurance Company, Hartford, Conn.:

William Pond Barber, Jr.; Thomas Kilburn Dodd; Gladstone Marshall; Leslie R. Martin; H. I. B. Rice.

Nominees of the National Life Insurance Company, Montpelier, Vt.:

Andrew John Blackmore; Austin Harris Hobson; Henry Holt; Henry Hollister Jackson; Clarence Edgeston Moulton.

Nominees of the Edison Electric Illuminating Company, Boston, Mass.:

Leonard L. Elden; Robert S. Hale; Harold Cyrle Hamilton; Sidney Hosmer; Irving E. Moultrop.

The Assistant Secretary announced the following elections by mail vote of the Council:

To sustaining membership:

The Aetna Life Insurance Company, Hartford, Conn.;
The American Life Insurance Company, Detroit, Mich.;
The Connecticut Mutual Life Insurance Company, Hartford, Conn.;
The Detroit Life Insurance Company, Detroit, Mich.;
The Eastman Kodak Company, Rochester, N. Y.;
The Edison Electric Illuminating Company, Boston, Mass.;
The Equitable Life Insurance Company of Iowa, Des Moines, Ia.;
The John Hancock Mutual Life Insurance Company, Boston, Mass.;
The Metropolitan Life Insurance Company, New York, N. Y.;
The National Life Insurance Company, Montpelier, Vt.;
The National Life Insurance Company of the United States of America, Chicago, Ill.;
The New England Life Insurance Company, Boston, Mass.

To ordinary membership:

Dr. Carl Louis Alsberg, Food Research Institute, Stanford University;
Miss Rose Lucile Anderson, Bryn Mawr College;
Mr. Frank Ayres, Agricultural and Mechanical College of Texas;
Mr. Lancelot Minor Berkeley, New York City;
Professor Durga Prasanna Bhattacharyya, Bareilly College;
Mr. Perry Aquila Caris, Philadelphia;
Mr. Edwin R. Carter, National Life Insurance Company, Chicago;
Professor William Frederick Durand, Stanford University;
Mr. James Drouie Flynn, Travelers Insurance Company, Hartford;
Mr. Thomas Henry Galbraith, Central Life Insurance Company of Illinois, Chicago;
Miss Marion Cameron Gray, Bryn Mawr College;
Mr. Leroy Francis Harza, Chicago;
Mr. Ray Nelson Haskell, Michigan Agricultural College;
Professor Mildred Hunt, Illinois Wesleyan University;
Dr. Carey M. Jensen, University of Wisconsin;
Professor Reginald Stevens Kimball, State Normal School, Worcester, Mass.;
Miss Helen Kunte, Hunter College;
Professor Anna Delia Lewis, Lake Erie College;
Professor John Hector McDonald, University of California;
Miss Laura Frances McDonough, Moravian College for Women;
Dr. Lee Horace McFarlan, University of Missouri;
Professor E. Loula McNeer, Morris Harvey College;

Professor Braxton Davis Mayo, Virginia Military Institute;
Mr. Earl Lewis Mickelson, Hamline University;
Professor Rayburn Z. Newson, Rusk College;
Mr. Frank Ordway, Phillips High School, Birmingham, Ala.;
Professor Delfin de la Paz, University of the Philippines;
Professor Carl John Rees, University of Delaware;
Mr. Charles Frederick Roos, Rice Institute;
Miss Elizabeth Thatcher Stafford, University of Texas;
Mrs. Ormelle Haines Stecker, Pennsylvania State College;
Mr. Henry J. Sternberg, Columbia University;
Professor Telesforo Tienzo, University of the Philippines;
Mr. William Arthur Watt, Pilot Life Insurance Company, Greensboro, N. C.;
Miss E. Kathryn Wyant, University of Missouri.

Nominees of Allyn and Bacon, Boston, Mass.:

Max G. Carman, University of Illinois; Alonzo Church, Princeton University; Orrin Frink, Columbia University; Raymond Garver, University of Chicago; John Haven Neelley, Yale University.

Nominees of the Connecticut General Life Insurance Company, Hartford, Conn.:

Walter Bjorn; Ward Van Buren Hart; Earl C. Henderson; Edward H. Hezlett; John Melvin Laird.

Nominees of the Eastman Kodak Company, Rochester, N. Y.:

Frederick E. Altman; F. M. Bishop; Charles Warnock Frederick; L. A. Jones; Ludwik Silberstein.

Phoenix Mutual Life Insurance Company, Hartford, Conn.:

Alden Thomson Bunyan; Henry W. Dewey; Henry N. Kaufman; Alson C. Patton; Harold Merle Springer.

Twenty-six applications for membership and thirty-two nominations by sustaining members were received.

The Council voted a resolution of thanks to Professor H. A. Perkins of Trinity College, Hartford, Conn., for the very valuable assistance he has rendered in connection with the endowment campaign, and to the Carnegie Corporation for a second contribution to the endowment fund.

The following appointments by President Birkhoff were announced:

To represent the Society at the Inauguration of Chancellor Benner at the University of Porto Rico, Professor C. G. P. Kuschke; to represent the Society at the Fiftieth Anniversary of the Founding of the George Peabody College for Teachers at Nashville, Tenn., Mr. Lewis C. Cox; to arrange for the next Josiah Willard Gibbs lecture, Professors E. R. Hedrick, Luby, E. H. Moore, Roever, and Stouffer

(Chairman); to consider the fees for colloquium lectures, Professors Archibald, H. H. Mitchell (Chairman), and Van Vleck; to consider the printing problems, Professors Cohen, Eisenhart (Chairman), Graustein, Hedrick, Slaught.

It was announced that the summer meeting and colloquium had been set for September 8–12, 1925, in Ithaca, N. Y.; that the Gibbs lecture in December, 1925, would be given by Professor J. Pierpont.

The Society decided to hold the Western Christmas meeting in Kansas City in conjunction with the meetings of the American Association for the Advancement of Science.

At the dinner held at the Del Prado Hotel on Friday evening, 67 persons were present. Professor Bliss presided and toasts were responded to by Professors E. H. Moore, Huntington, Kerékjártó, Stouffer, Birkhoff and Dresden. Professor Kerékjártó called attention to a new journal established in Hungary, to which further reference is made in the Notes in this issue.

On Saturday a resolution was passed thanking the Department of Mathematics of the University of Chicago for their hospitable reception.

The papers read at the meeting are listed below. Not mentioned in this list is the symposium address delivered on Friday afternoon by Professor W. D. MacMillan on *Some mathematical aspects of cosmology.* On Friday forenoon the Society met in two sections; the one for geometry, before which the papers numbered 1, 2, 3, 4, 5, 26, 27, and 30 were presented, was presided over by President Birkhoff; the other for algebra and theory of numbers, which heard the papers numbered 6, 7, 8, 9, 10, 11, 12, 13, 28, and 29, was presided over by Vice-President Hildebrandt. The latter also presided on Friday afternoon, while President Birkhoff took the chair during the closing session on Saturday forenoon.

The papers by Bennett, Caris, Reilly, Shohat, Wilson, and Ingraham's second paper were read by title; Mr. Jenkins

was introduced to the Society by Professor J. W. Glover, and Mr. Robinson by Professor E. P. Lane.

1. Professor R. M. Mathews: *Cubic curves and desmic surfaces.*

The twelve points of contact of the four tangents from each of three collinear points on a cubic of the sixth class form a Hessian $(12_4, 16_3)$ configuration on the curve. The sides of the three quadrangles of the points meet by threes in a similar configuration on a conjugate cubic which meets the given one in the nine vertices of the diagonal triangles of the quadrangles. A desmic surface is a quartic surface with twelve nodes which group into three tetrahedra each pair of which is perspective from each vertex of the third. These twelve nodes determine a counter-set of twelve nodes and for each point P of space there are two desmic surfaces, one on each set of nodes. The projection from P of the space configuration upon an arbitrary plane gives the configuration of the cubics. Various properties of the surfaces and curves are derived, among them relations with a 16_6 configuration and with a Poncelet hexagon and associated Steiner lines.

2. Professor Rufus Crane: *Another poristic system of triangles.*

This paper discusses the loci of points of a variable triangle having incircle and nine-point circle fixed. The loci of the centroid, circumcenter, orthocenter, and Nagel point are circles. The locus of the excenters is a limaçon; that of vertices of the triangle is a bicircular quartic. The relations of certain fixed points to these loci are also discussed.

3. Professor V. G. Grove: *A general theory of nets on a surface.*

In this paper, we first refer the sustaining surface to the asymptotic net, and then to an arbitrary non-conjugate net. We thus find that all of the projective properties of the net are expressible in terms of quantities characterizing the surface, and two arbitrary functions determining the most general non-conjugate net on the surface. From the forms of the formulas involved, one can tell at a glance those properties of the net which are really properties of the net, those which are properties of either of the two one-

parameter families of curves forming the net, and those which are properties entirely of the surface.

The R-reciprocal congruences associated with a net are Green reciprocals if and only if the tangents to the curves of the net form with the asymptotic tangents a constant cross ratio. There exists one and only one pair of Green reciprocal congruences that are in relation R with respect to a non-conjugate, non-asymptotic net.

4. Professor Arnold Emch: *On surfaces and curves which are invariant under involutory Cremona transformations.*

The present paper extends to projective spaces of n dimensions the investigations of the curves which are invariant under involutary Cremona transformations of two superposed planes begun in some papers in the TOHÔKU MATHEMATICAL JOURNAL.

If the transformation is given the form $\varrho x_i' = \varphi_i$, $i = 1, 2, \cdots, n+1$; and $\sigma x_i = \varphi_i(x')$, we form the Plückerian coordinates $\varrho p_{ik} = x_i\varphi_k - x_k\varphi_i$ of the line joining two corresponding points P and P'. These, as well as $\sigma q_{ik} = x_i\varphi_k + x_k\varphi_i$ are invariant under the transformation. By means of the identities $p_{ik}p_{jl} = q_{il}q_{jk} - q_{ij}q_{kl}$, it is found that in every involutory transformation in S_n there are two types of invariant surfaces (curves in S_2) representable by the equations $K(p_{12}, \cdots p_{ik} \cdots) = 0$, $L(q_{11}, q_{12}, \cdots, q_{ik}, \cdots, q_{ii}, \cdots) = 0$. When K is of even degree in the p_{ik}'s, then it may be classed with the form L. When K is of odd degree, then its square only belongs to the class L. Thus every invariant hypersurface, or its square, belongs to class L. Accordingly, invariant hypersurfaces belong to two classes: Those which belong to L but are not squares of K; those whose squares only belong to L. The second class is connected with the contact problem. As applications are given the quadratic transformation and its connection with the Cayley cubic surface, the Geiser transformation, transformations in S_3, the Schur sextic curve, the Enriques sextic surface, etc.

5. Dr. E. E. Libman: *Mean curvature curves on quadric surfaces.*

The name "mean curvature curve" is given to a curve (on a surface) at every point of which the mean curvature

of the surface is the same. This paper is an investigation into the properties of such curves on quadric surfaces. Some of the more interesting results follow. The mean curvature curves on a quadric are algebraic curves of order twelve. They touch both cyclic sections along a conic, the central section by a plane normal to the cyclic sections. Along a mean curvature curve the tangent of the angle at which two generators intersect upon the curve is proportional to the square of the distance of the tangent plane from the center. The mean curvature curve on a central quadric along which the mean curvature of the quadric is zero is the locus of the intersections of perpendicular pairs of generators. It is a sphero-conic upon the director sphere. The zero mean curvature curve on a paraboloid is a conic section, the intersection of a plane normal to the direction of the infinite center.

6. Professor J. F. Reilly: *Interpolation formulas containing parameters.*

An interpolation formula which gives satisfactory results throughout one portion of a table may yield rather inaccurate values throughout other portions of the same table. In this paper formulas are developed containing one or two parameters, which may be adjusted to fit any portion of the table. Sprague's third order differences osculatory interpolation formula is obtained by specializing the parameters. The use of such formulas is illustrated by showing how to adjust the parameters in order to obtain the best interpolated values in the least squares sense.

7. Professor A. A. Bennett: *On sets of three consecutive integers which are quadratic residues of primes.*

This paper appears in full in the present number of this BULLETIN.

8. Mr. P. A. Caris: *A solution of the quadratic congruence, modulo p, $p = 8n+1$, n odd.*

Explicit solutions of the quadratic congruence $x^2 \equiv a \pmod{p}$, where p is prime and of the form $4q+3$ or of the form $8n+5$ have been obtained by Legendre and others. The present paper contains explicit solutions for $a = -1, 2$ and n when $p = 8n+1$ and for a in general when n is odd.

9. Professor C. C. MacDuffee: *The nullity of a matrix relative to a field.*

The row-nullity ρ of a matrix x relative to a field F is defined as the number of linearly independent linear relations with coefficients in F among the rows of x; the column nullity $\varkappa$ is similarly defined. If a and b are two non-singular matrices with elements in F, then ρ and $\varkappa$ are invariants of x under the transformation $x' = axb$.

Certain relative projective invariants of algebraic forms written as determinants give rise to invariant matrices, e. g., the hessian matrix $H \equiv (f_{x_i x_j})$ of f_n is transformed according to the equation $H' = \breve{a}Ha$ by the projective transformation of matrix a. Thus the row-nullity of H relative to the field of the coefficients is an arithmetic invariant h of the form f_n itself. The complete solution of Hesse's problem is stated very simply in terms of h. Moreover, if the rank of H is 1, $h = n - 1$, and f_n is a constant times a perfect nth power of a linear function.

10. Dr. H. A. Bender: *On groups of order p^m which contain an abelian subgroup of order p^{m-1}.*

In every group of order p^m, p being an odd prime number, which contains an abelian subgroup of order p^{m-1} the abelian subgroup must be transformed in the same manner as it is transformed by an isomorphism of order p. In this paper we establish the necessary conditions that an isomorphism be of order p, and with this we proceed to show the possible ways a central and a commutator subgroup may be selected so as to give groups of order p^m and to determine the number of distinct groups there are for a given central and a given commutator subgroup.

11. Dr. H. A. Bender: *On orders of operators in the group of isomorphisms of prime power abelian groups.*

If $Q_n(x) = 0$ be the algebraic equation whose roots are the primitive nth roots of unity without repetition, then the group of isomorphisms of the abelian group of order p^m, p being a prime number, and type $(1, 1, 1, \cdots)$ contains operators of orders $p^\alpha \Pi_d Q_d(p)$ where d varies through all the distinct divisors of $a_1, a_2, \cdots, a_n$. The a's may be distinct and are to assume all possible values which satisfy the

relation $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n=\beta\leqq m\,(\beta=0,1,2,\cdots,m)$ $(p^{\alpha-1}<m-\beta\leqq p^{\alpha})$.

12. Professor G. A. Miller: *Imprimitive substitution groups.*

Various possible definitions of imprimitive substitution groups are observed, including the following: A necessary and sufficient condition that a transitive group G is imprimitive is that some co-set of G with respect to a subgroup G_1 composed of all the substitutions of G which omit a fixed letter, generates an intransitive group, or that such a co-set generates a proper subgroup of G. It is also necessary and sufficient that G contains a set of letters which does not include all its letters and is transformed into itself by all the substitutions of G which transform a fixed letter of G into some other fixed letter. To every proper subgroup generated by a co-set of G with respect to G_1, or by more than one such co-set, there corresponds a system of imprimitivity of G, and vice versa.

The number of possible sets of systems of imprimitivity of a transitive group G each of which involves two letters is always odd whenever there is a least set of such systems, and there is an infinite number of imprimitive groups which have the common property that each of them involves any arbitrary given odd number of different such systems of imprimitivity. In a transitive group whose order is of the form p^m, p being an odd prime, the number of the different systems of imprimitivity each of which involves exactly p letters is always of the form $kp+1$. This is not necessarily the case when the order of G is not of the form p^m. In fact, for every value of p it is possible to construct imprimitive groups which have exactly two different systems with just p letters in one system.

13. Professor G. E. Wahlin: *On a problem in Diophantine analysis.*

The author develops a method for obtaining the solutions of a Diophantine equation in which one member is a quadratic form in n variables and rational integral coefficients; and the other member a product of two or more unknown factors. It is based on the construction of an algebra over the field of rational numbers and consideration of the ideals of certain quadratic fields which are contained in the algebra.

14. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations.* Third paper, involving partial differential equations possessing a fundamental set of solutions.

There is a striking analogy between the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{2\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+a_1(x,y)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)+a_0(x,y)u=0,$$

where the a's are analytic near $(0,0)$, and an ordinary linear differential equation of the second order. By employing the operator $D=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$ as the analog of ordinary differentiation, one obtains a fundamental set of independent solutions such that the general solution may be written $u=\varphi_1 u_1+\varphi_2 u_2$. Here the φ's are functions of $x-y$. The initial conditions $u(x,0)=p(x)$, and $Du(x,0)=q(x)$, and analytic, determine a unique analytic solution. The theory extends to equations of higher order as well as to those of the first order. For the latter by Lagrange's method of solution one obtains a relation $y=f(x)-h$. With suitable boundary conditions the problem of expansion of an arbitrary function $f(x,y)$ involves a Birkhoff expansion along this curve and the coefficients are obtained by line integrals. The series converges to the mean value as illustrated by the expansion of xy in the case of the equation $Du+\lambda u=0$.

15. Professor K. P. Williams: *Non-synchronized relative invariant integrals.*

The paper deals with relative linear invariant integrals of the system $dz_i/dt=Z_i(z_1,\cdots,z_m)$. Such integrals are taken along closed paths drawn on a tube of trajectories. If the path passes through points that correspond to the same values of t we have a synchronized invariant $J=\int\sum L_i\delta z_i$. If it passes through points corresponding to different values of t we have a non-synchronized invariant $I=\int\sum L_i\delta z_i+K\delta t$. The paper develops a criterion on K so as to render I invariant if J is invariant.

16. Mr. J. A. Shohat: *On some properties of polynomials.*

In this paper the author considers integrals of the type $\int_a^b(x-a)^{\alpha-1}\ (b-x)^{\beta-1}q(x)\ M(x)\,dx\ (\alpha,\beta>0,\ q(x)\geqq q$ in

(a, b), $M(x)$ a polynomial; (a, b) finite). Using formulas previously established by the author in the theory of orthogonal Tchebycheff polynomials, he obtains some inequalities involving the maximum and minimum of $M(x)$, $M'(x)$ in (a, b); for example, a precise form of the law of the mean for polynomials. These results are applied, in particular, to polynomials which are monotonic in (a, b). A formula similar to one developed in this paper was given by Tchebycheff, using the theory of polynomials deviating the least from zero.

17. Professor H. L. Smith: *On functions of closest approximation.*

In the first part of this paper the notion of linear independence of n functions $p_1(x), \cdots, p_n(x)$, bounded in (ab), relative to a function $u(x)$ of limited variation on (ab) is defined. The existence of a linear combination $(c)(p) = c_1p_1 + \cdots + c_np_n$ for which the Stieltjes integral $\int_a^b |f-(c)(p)|^m\, du$ has its minimum value is then proved where f is a fixed function integrable (u) and m a fixed number $\geqq 1$. The minimizing function is proved to be unique when $m > 1$. A theorem is developed concerning the sequence of approximating functions corresponding to a convergent sequence of functions u and the theorem is applied to the study of the minimizing of finite sums.

In the second part the functions $p_1, \cdots, p_n$ are replaced by $(2n+1)$ trigonometric functions, the convergence of the approximating function to f as n approaches infinity is studied and the results are applied to the case of finite sums.

18. Professor Dunham Jackson: *On the convergence of certain processes of closest approximation over an infinite interval.*

This paper is concerned with polynomials affording the closest approximation to a given function over an infinite interval, according to the criterion of least squares (or least mth powers), with a weight function of appropriate character. Mr. Shohat (BULLETIN, Nov.–Dec., 1924) has discussed some of the problems relating to such polynomials of approximation. It is shown in the present paper that under specified hypotheses the approximating polynomial will converge to the value of the given function, as the degree of the polynomial is indefinitely increased. The

analysis applies in particular to the series of Hermite's polynomials used in the theory of statistics under the name of the Gram-Charlier series.

19. Professor Dunham Jackson: *On vector analysis in function space.* Preliminary communication.

This paper contains a further development of certain ideas which were presented for consideration at an earlier meeting of the Society (October, 1924). In particular, there is a discussion of a functional operator analogous to the divergence of a vector point function in ordinary space.

20. Professor W. H. Wilson: *Two related functional equations.*

The equation $f(x+y)=f(x)g(y)+g(x)f(y)+2\lambda f(x)f(y)$, where λ is a constant which may be zero, is discussed in part I of this paper and the functions, $f(x)$ and $g(x)$, satisfying this equation are expressed in terms of the functions $\varphi(x)$ and $\psi(x)$ whose addition theorems are $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\psi(x+y)=\psi(x)\,\psi(y)$, respectively. In part II the equation $g(x+y)=g(x)g(y)+\mu^2 f(x)f(y)$, where μ is a constant, is discussed and the functions, $f(x)$ and $g(x)$, satisfying this equation are expressed in terms of $\varphi(x)$ and $\psi(x)$.

21. Dr. R. W. Babcock: *On thermal convection.*

The differential equations which apply to slow steady convective motion of a viscous fluid are the standard equations of hydrodynamical theory, combined with equations expressing the law of thermal interchanges in a moving fluid. In the present paper approximate solutions are found following the method outlined by Oberbeck (ANNALEN DER PHYSIK, 1879, p. 271). A comparative study of the exact solution of a special system of non-homogeneous ordinary equations and its Oberbeck approximate solution shows that the magnitude of the constants in the system controls the rate of convergence of the approximation, and often produces divergence. Solutions with graphs of temperature and velocity fields have been computed for special cases of plane and cylindrical boundaries with sinusoidal impressed boundary temperatures. If this temperature has a vertical gradient only, the Oberbeck method produces a null solution, corresponding to equilibrium of the fluid in stratified layers. Arithmetical solutions for common liquids at ordinary

temperatures show that the Oberbeck solutions diverge unless the gradient is much too small to be easily applied in laboratory practice.

22. Professor H. W. March: *The deflection of a rectangular plate with two opposite edges supported and two edges free.*

Solutions of this problem for the case of a uniform load and of other problems relating to the rectangular plate with two opposite edges supported were given by Estanave (ANNALES DE L'ECOLE NORMALE, 1900) using the method of Lévy. This paper gives a simpler solution of the problem for the case of a uniform load and also a solution for the case of a load which is a function of x alone, x being measured in the direction perpendicular to the supported edges. The simplification effected results from first choosing a solution of the non-homogeneous differential equation which satisfies part of the boundary conditions and then adding to this a suitably chosen solution of the corresponding homogeneous equation.

23. Professor M. H. Ingraham: *Solution of certain functional equations relative to a general number system.*

Relative to any normally-orderable associative division number system $\mathfrak{A}$, there exists a range P and a field F, either the field of rationals or a field of integers modulo a prime, such that $\mathfrak{A}$ is isomorphic with the set V of all finitely non-zero functions of P to F. Relative to functions on such a system V to V it is shown that any solution of the functional equation

$$\sum_{i}^{0,n+1} (-1)^{i}_{n+i} c_i f(v_0 + i v_1) = 0 \qquad (v_0, v_1)$$

is, if F is rational or has a modulus $n+1$, a polynomial of degree equal to or less than n in the coordinates of the elements of V with coefficients in V and conversely. The trivial case in which F has a modulus $m \leq n+1$ is dealt with. The paper is an extension of the results of Hamel for the equation $f(x+y) = f(x) + f(y)$ and the real number system. Hamel made use of the Zermelo "axiom of selection".

24. Professor E. V. Huntington: *Postulates for order on a closed line*: I. *Reversible order (separation of point-pairs).*

The theory of reversible order on a closed line (also known as the theory of separation of point-pairs), which is of fundamental importance in the development of projective geometry, was first systematically studied by Pasch in 1882. The first set of postulates for this theory was given by Vailati in 1895, and has been substantially followed by later writers. The present paper gives a more exhaustive analysis of the properties of this type of order (expressed in terms of a tetradic relation *ABCD*), and leads to the following new set of six postulates (of which the first two serve only to exclude obviously trivial cases):

0. There is at least one true tetrad, say *XYZW*.
00. If *ABCD* is true, then the elements *A, B, C, D* are distinct.
I. If *ABCD* is true, then *BCDA* is true. (Cyclicality.)
II. If *ABCD* is true, then *ABDC* must be false. (Homogeneity.)
III. If *ABCD* is true, and *X* is any fifth element, then at least one of the relations *AXCD* and *ABCX* is true. (Connexity.)
IV. If *ABCD* is true, then *DCBA* is true. (Reversibility.)

This paper (which extends and completes an unpublished paper presented to the Society in December, 1916) will be followed by a second paper on irreversible order, in which Postulate IV is replaced by its opposite.

25. Professor J. B. Shaw: *On the classification of linear algebras.*

The classification of linear algebras, whether associative or not, may be made from two very different standpoints. In one the basis is that of the properties of any number, the general number, considered as held fixed, while all other numbers are multiplied by it. This would correspond geometrically to a collineation. The classification on this basis I have carried out in previous papers. A different basis is that initiated by Cayley in his paper on double algebras, and is based upon the numbers that are invariant in the algebra in the sense of being their own squares, or are zero. That is they are idempotent or nilpotent. The latter basis is geometrically that of quadratic transformations. In both cases there are invariants that must count double, triple, etc., and the number of these furnishes the basis for the divisions. In the paper a study is made of the conditions this multiplicity imposes upon the system. It

shows up in the formation of certain sub-algebras. Non-associative algebras may be classified from either standpoint, the property of associativity becoming relatively unimportant.

26. Professor Louis Ingold: *Associated types of linear connection.*

In this paper the transformation $\overline{\Gamma}^s_{ik} = A^s_\alpha\left(\frac{\partial a^\alpha_i}{\partial u^\lambda} + a^\tau_i \Gamma^\alpha_{\tau\lambda}\right)$ on the functions Γ^k_{ij}, which are used to define the neighborhood connections in a geometry, is associated with an arbitrary matrix $||a^i_j||$. The elements a^i_j are functions of the coordinates of the space and $A^s_\alpha \cdot |a^i_j|$ is the cofactor of a^s_α.

It is shown that the resultant of the transformations associated with two matrices is again a transformation of the same type,—in fact, the one associated with the resultant matrix. The geometries corresponding to different sets of functions Γ related in this way are called associated geometries. The effect of the transformations on certain fundamental tensors is considered and a series of properties is found which hold in all associated geometries.

27. Professor Ingold: *The geometry of a set of n vectors.*

A set of n independent vectors is taken as the basis of a geometry. For convenience they are represented in a function space by the functions $\theta_i(x; u^1, u^2, \cdots, u^n), i = 1, 2, \cdots, n$.

Quantities analogous to the fundamental quantities and Christoffel symbols of differential geometry are formed from the functions θ_i and their derivatives with respect to the coordinates u^i. These serve to define the neighborhood connections. If instead of the functions θ_i, a set of n independent functions φ_j are used as basis, where $\varphi_j = a^i_j \theta_i$, the coefficients Γ^k_{ij} of the linear connection undergo the transformation discussed in the previous paper.

28. Mr. W. A. Jenkins: *On a central difference summation formula.*

Three finite difference formulas are discussed each of which interpolates a certain number of values between each of a given series of equidistant ordinates and sums approximately the interpolated along with the given values, in terms of the original ordinates and their differences. Two of these, due to Lubbock and De Morgan, are proved inferior in convergency to the third, a central difference

formula derived by Woolhouse in the JOURNAL OF THE INSTITUTE OF ACTUARIES, volume 11, page 307. An approximate test of accuracy is developed.

29. Professor J. S. Turner: *Note on prime factors.*

This note presents a short and direct proof of the theorem "If a prime p is a factor of a product ab, where a, b are positive integers and a is prime to p, then p is a factor of b."

30. Mr. P. G. Robinson: *Surfaces with constant absolute invariants.*

Consider a surface such that the homogeneous coordinates of any point of it are analytic functions of two parameters. Let the asymptotic tangents of any point of the surface be distinct, and let neither have more than second order contact with the surface at any point. Wilczynski has shown that under these conditions and by a proper choice of the tetrahedron of reference, the surface may be represented by a development of the form

$$z = xy + \frac{1}{6}(x^3 + y^3) + \frac{1}{24}(Ix^4 + Jy^4) + \cdots$$

where I and J are absolute invariants of the surface.

This paper shows first that there are no surfaces for which either I or J is equal to zero and the other equal to a constant different from zero, at all points which satisfy the conditions formulated above. Then the case is considered where both I and J are equal to non-vanishing constants. In this case the differential equations of the surfaces are obtained and integrated. From these results are obtained properties of the surfaces.

31. Professor M. H. Ingraham: *A general theory of linear sets.*

In this paper the author, using the postulational method, considers classes of vectors, on a finite range, whose elements belong to a general division algebra, and gives a general basis preliminary to the more intensive study of classes of vectors on a general range and of number systems over a division number system. This paper has appeared in the April, 1925, number of the TRANSACTIONS OF THIS SOCIETY.

ARNOLD DRESDEN,
Assistant Secretary.

CONCERNING THE COMPLEMENTARY INTERVALS OF COUNTABLE CLOSED SETS*

BY J. R. KLINE

Professor R. L. Moore† recently gave an example showing that the following statement due to Hobson‡ is incorrect:

"*A non-dense closed set is enumerable if its complementary intervals are such that every one of them abuts on another one at each of its ends.*"

In considering this example, we notice that while each of the complementary intervals of the original set of Moore's example are such that each one of them abuts on another one at each of its ends, the same is not true if we consider the complementary intervals of G', the first derived set§ of G. This leads us to the following theorem:

THEOREM A. *A non-dense closed set G is enumerable, if G and every derived set of G has the property that every one of its complementary intervals is such that it abuts on another one at each of its ends.*

PROOF: Let us suppose that G is not enumerable. Then there exists a number β of the first or second class such that the derived set $G^{\beta+1}$ is identical with the derived set G^{β} and G^{β} is perfect.|| It follows that G^{β} has no

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* Presented to the Society, October 25, 1924.

† See R. L. Moore, *An uncountable, closed, and non-dense point set, each of whose complementary intervals abuts on another one at each of its ends*, this BULLETIN, vol. 29, (1923), pp. 49–50.

‡ See E. W. Hobson, *The Theory of Functions of a Real Variable* 1st edition (1907), p. 92; and 2d edition (1921), p. 113.

§ If G is a point set, then the set G' consisting of all limit points of G is called the *derived set* or the derivative of G.

|| See E. W. Hobson, loc. cit., p. 115. In his proof of the theorem that every non-dense linear closed set is, in general, made up of an enumerable set and a perfect set, Hobson here shows that unless the non-dense linear closed set G is enumerable, there is a number β of the first or second class such that G^{β} is perfect.

isolated points. As G is non-dense and closed, it follows that G^β is non-dense. Select a complementary interval of G^β. As this interval abuts on another complementary interval of G^β at both its ends, it follows that the supposition that G is not enumerable has lead to a contradiction.

As Hobson points out, it is not necessary that each complementary interval abut on another at each of its ends in order that a set be enumerable. But as no subset of a closed countable set is perfect,* then there must be some definite number β of the first or second class such that $G^{\beta+1}$ vanishes. Thus G^β consists only of isolated points. It follows that its complementary intervals abut at both their ends. Thus we are enabled to state the following theorem.

THEOREM B. *In order that a linear closed set be enumerable it is both necessary and sufficient that there exist some number β of the first or second class such that G^β and all the derived sets of G after G^β have the property that each of its complementary intervals abuts on another one at both its ends.*

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* See R. L. Moore, *An extension of the theorem that no countable point set is perfect*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 10. pp. 168–170.

ON SETS OF THREE CONSECUTIVE INTEGERS WHICH ARE QUADRATIC RESIDUES OF PRIMES*

BY A. A. BENNETT

In this paper we shall prove the following theorems.

THEOREM I. *For each prime, p, for which there are as many as three incongruent squares, there is a set of three consecutive residues (admitting zero and negative numbers as residues) which are squares, modulo p.*

THEOREM II. *For $p = 11$, and for each prime p greater than 17, (and for no other primes), there is a set of three consecutive least positive (non-zero) residues which are squares, modulo p.*

The problem† of finding three consecutive integers which are quadratic residues of a prime, p, is equivalent to the formally more general problem of finding two quantities, x, y, $(y \not\equiv 0)$, such that x, y, $x+y$, $x-y$, are proportional to squares in the domain,‡ since we then have $(x/y)-1$, x/y, $(x/y)+1$ as consecutive squares in the domain. We may show that for residues with respect to a modulus the condition is equivalent to the existence of a square of the form§ $u\,v\,(u+v)\,(u-v)$. By taking $u = x$, $v = y$, we see that the condition is necessary.

* Presented to the Society, April 10, 1925.

† For references, compare article of similar title by H. S. Vandiver, this BULLETIN, vol. 31 (1925), p. 33.

‡ That, in the system of natural numbers, it is impossible to have distinct quantities, x, y, such that x, y, $x+y$, $x-y$ are all proportional to squares was proved by Fermat by his celebrated method of "infinite descent". See Carmichael, *Theory of Numbers*, p. 86.

§ It is of interest to note that in the case of natural numbers we may take $u = x$ and $v = y$ for this relation. Indeed, if $x, y, x+y, x-y$ were proportional to squares, certainly their product would be a square. Conversely, suppose that their product were a square. Then either x, y, $x+y$, $x-y$ would all be relatively prime, or if

That it is also sufficient may be shown as follows. Take $x = (u^2+v^2)^2$ and $y = 4uv(u^2-v^2)$. Now x may also be written in the form $4u^2v^2+(u^2-v^2)^2$. Hence $x+y = [2uv+(u^2-v^2)]^2$, and $x-y = [2uv-(u^2-v^2)]^2$. Thus if $uv(u^2-v^2)$ is a perfect square, so also are x, y, $x+y$, and $x-y$, when these are related in this manner.

The expression $uv(u+v)(u-v)$ takes on the values $6^2\cdot 5$, $2^2\cdot 6$, $2^2\cdot 30$, for the choices of (u, v) as (5,4), (3,1), (5,1), respectively. But at least one of the three numbers 5, 6, 30 is a quadratic residue of the prime p no matter how p is chosen. Hence there is always a choice of $uv(u+v)(u-v)$ which is a non-zero quadratic residue for each prime p greater than 5. The corresponding solutions of the original problem are (1/4, 5/4, 9/4), (1/24, 25/24, 49/24), (49/120, 169/120, 289/120). These numbers in turn are all different from zero for $p = 11$, or for $p > 17$, but not otherwise. Now every three consecutive residues no one of which is congruent to zero are congruent to a set of three consecutive least positive (non-zero) residues. Thus we establish the theorems announced.

There is no difficulty in obtaining linear forms, the primes within which are such that for each of these choices of p, a preassigned number of consecutive residues shall be squares. Indeed, we have merely to select assigned numbers to be quadratic residues. Thus for p of the form $24k+1$ or $24k+23$, each of the numbers 1, 2, 3, 4 is a square. By choosing a form for which 2, 3, 5, 7, -1 are all quadratic residues, and dropping the condition of positiveness, we have always the following twenty-one consecutive residues as squares, -10, -9, -8, $\cdots$, 9, 10.

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two had a common factor, d, this would be a factor throughout. Write $x = dx'$, $y = dy'$; then $x'y'(x'+y')(x'-y')$ also would be a square. Its four relatively prime factors would then be individually squares and x, y, $x+y$, $x-y$ would therefore be proportional to squares. This method of argument is inapplicable to the case of residues.

GROUPS IN WHICH THE NORMALISER OF EVERY ELEMENT EXCEPT IDENTITY IS ABELIAN*

BY LOUIS WEISNER

1. *Introduction.* Certain groups, of which the tetrahedral and icosahedral groups are non-trivial examples, possess the property that the normaliser of every element except identity is abelian. In this paper we shall investigate the properties of such groups.

2. *Abelian Subgroups.* An abelian subgroup of a group will be called a *maximal* abelian subgroup if it is not contained in any larger abelian subgroup of the group.

If two abelian subgroups of a group G, in which the normaliser of every element except identity is abelian, have an element besides identity in common, they generate an abelian group; for an element common to the abelian subgroups H and I is invariant under (H,I), which must therefore be abelian. It follows that *the maximal abelian subgroups of G are independent*; that is, no two maximal abelian subgroups of G have an element in common besides identity.

Since every prime-power group possesses an invariant element besides identity, *the Sylow subgroups of G are abelian.* Moreover, *the Sylow subgroups of G are independent.* For if two Sylow subgroups H and I (necessarily of the same order p^a) had an element in common besides identity, (H,I) would be an abelian subgroup of G of order $p^b > p^a$, which contradicts the fact that p^a is the highest power of p that divides the order of G.

Let H be a maximal abelian subgroup of G, I a Sylow subgroup of H, and J that Sylow subgroup of G which includes I. Since H and J have I in common, (H,J) is

* Presented to the Society, September 8, 1925.

an abelian group. But H is a *maximal* abelian subgroup of G. Hence H contains J. Therefore *the Sylow subgroups of a maximal abelian subgroup of G are Sylow subgroups of G.* It follows that *the orders of two non-conjugate maximal abelian subgroups of G are relatively prime.* If G possesses exactly r conjugate sets of maximal abelian subgroups, of which $H_1, H_2, \cdots, H_r$, of orders $h_1, h_2, \cdots, h_r$ respectively, are representatives, then every element of G except identity is included in one and only one conjugate of one of the H's, and the order of G is $h_1 h_2 \cdots h_r$, $h_1, h_2, \cdots, h_r$ being relatively prime.

If a maximal abelian subgroup H, of order h, has y conjugates under G, the total number of elements of G whose orders divide h is $y(h-1)+1$; for every element whose order divides h is included in a conjugate of H, and the conjugates of H are independent. By Frobenius' theorem,

$$y(h-1)+1 \equiv 0, \pmod{h},$$

whence

$$y \equiv 1, \pmod{h}.$$

Therefore, *the number of conjugates of a maximal abelian subgroup of order h is of the form $1+xh$.*

3. *Solvable Groups.* Let G be a solvable group in which the normaliser of every element except identity is abelian. A minimal invariant subgroup I is an abelian group. The maximal abelian subgroup H which contains I must be invariant under G; otherwise I would be included in H and a conjugate of H, whereas the maximal abelian subgroups of G are independent.

Arrange G in cosets as regards H. Every element of G transforms H into itself, thus establishing an isomorphism of H. Elements in the same coset establish the same isomorphism, while elements in different cosets establish different isomorphisms. Since no element of H except identity is commutative with an element of G not in H, G/H is simply isomorphic with a group of isomorphisms of H, each of which leaves no element of H unchanged

except identity. It follows* that G/H is either a cyclic group or a dicyclic group. If G/H is dicyclic, it contains a non-abelian Sylow subgroup of even order and G contains a subgroup simply isomorphic with this Sylow subgroup, since the order of H is prime to the order of G/H. But the Sylow subgroups of G are abelian. Hence G/H is a cyclic group.

Thus *a solvable group in which the normaliser of every element except identity is abelian contains an invariant maximal abelian subgroup, and the corresponding quotient group is cyclic.*

4. *Insolvable Groups.* An insolvable group G in which the normaliser of every element except identity is abelian must contain a minimal invariant subgroup H, which is a simple group of composite order. If G is larger than H, let I be a subgroup of G which contains H, such that H is of prime index p in I.

Suppose that no element of H is commutative with an element of I which is not in H. A maximal abelian subgroup J of H must also be a maximal abelian subgroup of I; otherwise, H would contain two non-conjugate abelian subgroups of the same order, one of which is a maximal abelian subgroup, which conflicts with a theorem proved in § 2. Therefore the normaliser K of J in I is larger than, and includes, the normaliser L of J in H. By § 3, K/J is a cyclic group. Since J is a maximal abelian subgroup of K, the order of J is prime to the order of K/J. Hence K contains an element t whose order n equals the order of K/J, and $K = (t, J)$. Since L is a subgroup of K which includes J, $L = (t^a, J)$, where a is a divisor of n. Since t^a is an element of H which is commutative with t, $t^a = 1$, and $L = J$. It follows that J is the normaliser in H of every Sylow subgroup of J. Since the Sylow subgroups of J are Sylow subgroups of H, H is not a simple group,† contrary to supposition.

* Burnside, *Theory of Groups*, 2d edition, p. 336, Theorem V.

† Burnside, p. 327, Theorem II.

It follows that I contains an element t not in H, which is commutative with some element of H other than identity. Let J be the normaliser of t in I, and K the subgroup common to J and H. Since the maximal abelian subgroups of H and of I are independent, while J and K are maximal abelian subgroups of I and H respectively, the number of conjugates of J under I equals the number of conjugates of K under H. Let this number be x, and let h, i, j, k denote the orders of H, I, J, K respectively. The elements of H whose orders divide $j = kp$ are included in the conjugates of K, while the elements of I whose orders divide kp are included in the conjugates of J. Observing that the order of every element of I not in H divides kp, we have

$$x\,(kp-1)+1 = x\,(k-1)+h\,(p-1),$$

from which $x = h/k$. Therefore K is the normaliser of K in H. It follows as in the preceding paragraph that H is not a simple group.

Since the assumption that G is larger than H leads to a contradiction, it follows that *an insolvable group in which the normaliser of every element except identity is abelian is a simple group.*

The linear fractional group of order $2^n(2^{2n}-1)$, $(n>2)$ is an example of a simple group in which the normaliser of every element except identity is abelian. That this group actually possesses the property in question follows from its analysis.*

UNIVERSITY OF ROCHESTER

* Dickson, *Linear Groups*, pp. 262–265.

NOTE ON GIBBS' PHENOMENON

BY C. N. MOORE

In my review of the third edition of volume I of Picard's *Traité d'Analyse*,* I took exception to Picard's claim that Du Bois-Reymond had discovered Gibbs' phenomenon. Concerning the question of priority in this discovery, Professor G. N. Watson has kindly called my attention to an article by H. Wilbraham.†

The phraseology of this paper indicates that Wilbraham was not quite clear in his own mind as to the distinction between the curve which is the limit curve of the approximation curves, $y = S_n(x)$, and the curve $y = f(x)$, which represents the limit function. However, his discussion applies to the former curve and furnishes a valid proof of the central features of Gibbs' phenomenon in the case of the particular series he discusses. He also points out that a similar discussion will establish an analogous behavior in the case of another special series. As Professor Watson has said in his letter to me: "It is a remarkable paper for so early a date."

At first thought it seems rather surprising that Wilbraham's paper, dealing with so interesting a property of Fourier's series, should have remained virtually unnoticed for a period of more than half a century, during which the theory of these series was being greatly enlarged. There is, however, a rather natural reason for this, the same reason why Gibbs' paper in NATURE at first attracted no general attention. Both Wilbraham and Gibbs restricted their discussion to particular series, and failed to point out that the property in question characterized the behavior of the Fourier's series corresponding to a very broad class of

* This BULLETIN, (2), vol. 30 (1924), pp. 554–556.

† CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL, vol 3 (1848), pp. 198–201.

functions. It was Bôcher's wide generalization* of Gibbs' brief remarks, as well as his lucid exposition of the details of Gibbs' phenomenon, that first aroused general interest in this remarkable property of Fourier's series.

Subsequent to writing the preceding paragraphs I received a letter from Professor H. S. Carslaw, calling my attention to Wilbraham's paper, and I also had the privilege of seeing a copy of the manuscript of Professor Carslaw's historical note, *Gibbs' phenomenon in Fourier's series and integrals.*† Since this note goes into some detail concerning Du Bois-Reymond's paper, it seems to me desirable to amplify somewhat my own position regarding this paper, as previously indicated in the review to which reference has been made.

The formulas which Professor Carslaw quotes from Du Bois-Reymond's paper are the ones to which I referred when I made the statement that "there are certain formulas in the article from which Gibbs' phenomenon might have been deduced". However, if one goes back to page 244 of Du Bois-Reymond's article, the paragraph beginning at the bottom of this page reveals the fact that he had decided on the basis of physical intuition as to how the approximation curves ought to behave. His analysis seems to have been carried out with a view to substantiating this prejudgment on his part, and that fact in my opinion explains his error as to the behavior of the integral $\int_0^{n\pi} [(\sin \alpha)/\alpha]\, d\alpha$. If he had examined the behavior of this integral on its own merits, it does not seem likely that he would have made such an error. But the extreme limits $\pm(\pi/2)$ fitted in with his preconceived ideas, so he apparently jumped to the conclusion that they were correct, having in mind doubtless the relation $\int_0^{+\infty} [(\sin \alpha)/\alpha]\, d\alpha = \pm(\pi/2)$.

It seems to the writer that Gibbs' phenomenon is most decidedly one of those facts that can only be discovered by the methods of analysis. The approximation curves

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* Annals of Mathematics, (2), vol. 7 (1905-06), pp. 123-132.

† See page 419 of the present number of this Bulletin.

$y = S_n(x)$ are obviously continuous curves, and either geometric or physical intuition would naturally suggest that they might behave as Du Bois-Reymond supposed. It is only the more delicate considerations of analysis that would reveal the true facts. That so competent an analyst as Du Bois-Reymond should have overlooked these facts, when he had formulas before him from which they could readily have been deduced, seems to me all the more reason for not associating his name with the phenomenon in question. It is quite natural and proper to refer to his work in giving the historical background of Gibbs' phenomenon, but in my opinion Professor Picard's designation of the phenomenon as "Phénomène de Du Bois-Reymond et Gibbs" was without adequate justification, even if Wilbraham's article had not existed.

THE UNIVERSITY OF CINCINNATI

A HISTORICAL NOTE ON GIBBS' PHENOMENON IN FOURIER'S SERIES AND INTEGRALS

BY H. S. CARSLAW

In 1899, Gibbs called attention* to the fact that for large values of n the approximation curves

$$y = S_n(x) = 2\sum_1^n (-1)^{r-1}\frac{\sin rx}{r},$$

for the Fourier's series which represents $f(x) = x$ in the interval $-\pi < x < \pi$, fall from the point $(-\pi, 0)$ at a steep gradient to a point very nearly at a depth $2\int_0^\pi [(\sin\alpha)/\alpha]d\alpha$ below the axis of x, then oscillate above and below $y = x$ close to this line until x approaches π, when they rise to a point very nearly at a height $2\int_0^\pi [(\sin\alpha)/\alpha]d\alpha$ above the axis, and then fall rapidly to $(\pi, 0)$.

At the point of discontinuity, where $x = \pi$, in the series $2\sum_1^\infty (-1)^{r-1}(\sin rx)/r$ the approximation curves thus tend to coincide, not with the segment joining the points (π, π) and $(\pi, -\pi)$, but with the straight line whose ends are the points

$$\left(\pi, \pi + \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right)$$

and

$$\left(\pi, -\pi - \frac{D}{\pi}\int_\pi^\infty \frac{\sin\alpha}{\alpha}\,d\alpha\right),$$

where $D = f(\pi+0) - f(\pi-0)$, the amount of the "jump" in the sum of the series at that point.

In 1906, Bôcher showed† that the same phenomenon occurred in general in the Fourier's Series for the arbitrary

* NATURE, vol. 59 (1899), p. 606.

† ANNALS OF MATHEMATICS, (2), vol. 7, (1906).

function $f(x)$ in the neighborhood of a point of discontinuity whose abscissa is a, when $f(a \pm 0)$ exist. The approximation curves approach a limiting curve in which the vertical portion is the line joining the points $(a, f(a-0)$ and $(a, f(a+0))$; *but this vertical line has to be produced beyond these points by an amount that bears a definite ratio to the magnitude of the jump.*

Bôcher gave the name *Gibbs' phenomenon* to this property of the approximation curves of the Fourier's series. Both Bôcher and Gibbs were under the impression that the property had remained unsuspected till Gibbs discovered it for the series

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

in 1899.

In my book on *Fourier's Series and Integrals*,* I repeat this statement, and in the review of the third edition of Picard's *Traité d'Analyse* by C. N. Moore,† it is again to be found in print. *However, it is now known that Gibbs's phenomenon had been observed about* 50 *years earlier than* 1899. Indeed, in 1848, Wilbraham,‡ a B. A. of Trinity College, Cambridge, discussed the equation

$$y = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots.$$

His object was to show that this equation represents "a locus composed of separate straight lines, of which each is equal to π, parallel to the axis of x, and at a distance $\pm \frac{1}{4}\pi$ alternately above and below it, joined by perpendiculars which are themselves part of the locus" but extend beyond the ends $\pm \frac{1}{4}\pi$ by a distance $\frac{1}{2}\left|\int_{\pi}^{\infty}[(\sin \alpha)/\alpha]d\alpha\right|$ at either extremity.

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* Carslaw, *Fourier's Series and Integrals*, 2d ed., 1921, p. 266.

† This Bulletin, vol. 30 (1924), p. 555.

‡ *On a certain periodic function*, Cambridge and Dublin Mathematical Journal, vol. 3 (1848), p. 198.

It could not be expected that Wilbraham in 1848 would have a clear idea of what is meant by the sum of the series for the value x_0: i. e., in our notation $\lim_{n\to\infty} S_n(x_0)$. In dealing with this sum he allows x to be a function of n which tends to x_0 as $n\to\infty$; he takes the sum $S_n(x)$ for the positive integer n and the corresponding value of x_0 and he then lets n tend to infinity. But, as his diagrams show, he saw quite clearly the behavior of the approximation curves of his series of cosines, and he remarked that "a similar investigation of the equation

$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

would lead to an analogous result".

This second series, it will be seen, is the same as that which Gibbs used in 1899.

Wilbraham's paper seems to have attracted little attention and to have been forgotten until Burkhardt quoted it in his article on *Trigonometrische Reihen und Integrale* (*bis etwa* 1850).* The fact that Gibbs' phenomenon had been observed earlier is mentioned also by Hilb and Riesz in their recently published article *Neuere Untersuchungen über trigonometrische Reihen.*†

We may still call this property of Fourier's series (and certain other series) Gibbs' phenomenon; but we must no longer claim that the property was first discovered by Gibbs.

In the review of Picard's *Traité d'Analyse* mentioned above, Moore comments on the title which Picard gives to the section in which Gibbs' phenomenon is discussed, namely, "*Phénomène de Du Bois-Reymond et Gibbs*". He concludes that it is wrong to associate Du Bois-Reymond's name with this discovery. In this I think the reviewer is right. But it is certainly true that, if Du Bois-Reymond

* Encyklopädie der Mathematischen Wissenschaften, Bd. II, Teil I, 2. Hälfte, p. 1049, 1914.

† Encyklopädie, Bd. II, Teil 3, p. 1203, 1924.

had not made a curious slip as to the values which the integral $\int_0^{nx}[(\sin\alpha)/\alpha]d\alpha$ can take when $n\to\infty$ and $x\to 0$ simultaneously, he would have enunciated Gibbs' phenomenon both for Fourier's series and Fourier's integral in 1874.

The memoir* to which Picard refers is well known. In it he distinguishes carefully between

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} f(x\pm\varepsilon, n),$$

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} f(x\pm\varepsilon, n),$$

and the case when n tends to infinity and ε to zero simultaneously: e. g. $\varepsilon=\varphi_1(t)$ and $1/n=\varphi_2(t)$, while as t tends to its limit, $\varphi_1(t)$ and $\varphi_2(t)$ tend to zero.

He uses the symbol Lim for such an arbitrary simultaneous approach of n to infinity and ε to zero, and his work is based on the second theorem of mean value. But even when his results are correct, the argument is not always clear or rigorous.

Yet he finds (loc. cit. § 9) that in the case of Fourier's integral, when $x\to x_1$ and $h\to\infty$, we have

$$\operatorname{Lim}\frac{1}{\pi}\int_0^h d\alpha\int_{-\infty}^{\infty} d\beta\, f(\beta)\cos\alpha(\beta-x)$$

$$=\frac{1}{2}\{f(x_1+0)+f(x_1-0)\}$$

$$+\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\operatorname{Lim}\int_0^{h(x-x_1)}\frac{\sin\alpha}{\alpha}d\alpha,$$

while for Fourier's series, when $x\to x_1$, and $n\to\infty$, we have

$$\operatorname{Lim} S_n(x) = {}^1/_2\{f(x_1+0)+f(x_1-0)\}$$

$$+\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\operatorname{Lim}\int_0^{n(x-x_1)}\frac{\sin\alpha}{\alpha}d\alpha.$$

* *Über die sprungweisen Werthänderungen analytischer Functionen*, Mathematische Annalen, vol. 7 (1874), p. 241.

But he goes on to say that the integrals

$$\int_0^{h(x-x_1)} \frac{\sin\alpha}{\alpha}\,d\alpha \text{ and } \int_0^{n(x-x_1)} \frac{\sin\alpha}{\alpha}\,d\alpha,$$

which occur in these results, can take every value between $\pm\frac{1}{2}\pi$— of course this should have been $\pm\int_0^{\pi}[(\sin\alpha)/\alpha]d\alpha$ — according to the way in which h and $(x-x_1)$, or n and $(x-x_1)$, tend to infinity and zero respectively. He thus concludes that the Fourier's integral and series at a point of ordinary discontinuity, in this simultaneous limit, represent all the values between $f(x_1-0)$ and $f(x_1+0)$, instead of extending beyond $f(x_1\pm 0)$ by the amount

$$\left|\left\{\frac{f(x_1+0)-f(x_1-0)}{\pi}\right\}\int_{\pi}^{\infty} \frac{\sin\alpha}{\alpha}\,d\alpha\right|.$$

Further, Du Bois-Reymond seems to have been the first to bring up the question of the behavior of Fourier's integral in this connection. And with a corresponding emendation his results are correct.

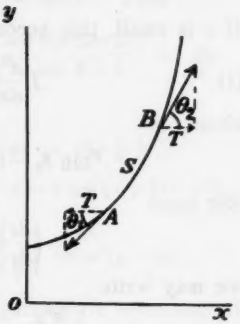
Early in 1925, I sent to the QUARTERLY JOURNAL OF MATHEMATICS a paper on Gibbs' phenomenon in one of the early sections of which I refer to the work of Wilbraham and Du Bois-Reymond. But the mistake about the discovery of Gibbs' phenomenon is so common that it seems proper to make this correction also in this BULLETIN.

SYDNEY, AUSTRALIA

A GENERAL FORM OF THE SUSPENSION BRIDGE CATENARY

BY IRA FREEMAN

1. *Introduction.* The form of the curve assumed by a uniform chain or string hanging freely between two supports was first investigated by Galileo, who erroneously determined it to be a parabola; Jungius detected Galileo's error, but the true form was not discovered until 1691, when James Bernoulli published it as a problem in the ACTA ERUDITORUM. He also considered the cases when the chain was (1) of variable density, (2) extensible, (3) acted upon at each point by a force directed to a fixed center.



These curves attracted much attention, and were discussed by John Bernoulli, Leibnitz, Huygens, David Gregory and others. A review of the literature shows, however, that in the problem of the inextensible string acted upon by gravitational forces, only two special cases are considered:—the case where there is simply a string of uni-form linear density, and the case of a string of inappreciable weight supporting a uniform horizontal load. Moreover, the modern text-books in the subject appear to shun a more general case, and content themselves with presenting only the two very special cases mentioned.

The object of the present paper is to consider what happens when a cable of appreciable uniform linear density supports a uniform horizontal load, conditions that are very nearly realized in the suspension-bridge. The texts solve the special cases by the solution of two first-order

differential equations, the first of which gives the intrinsic equation of the curve. The following method uses one second-order differential equation, and does not involve the intrinsic equation as an intermediate step.

2. *Densities Different and not Zero. Both the cable and the bridge platform have uniform linear densities which are appreciable, and, in general, different.*

Let α be the weight per unit length of the cable, and let β represent that of the bridge platform. If any arc s is in equilibrium, the horizontal component of the tensions, at the extremities of the arc, will be the same constant which we may call T. The difference of the vertical components of the extreme tensions will equal the total weight on the arc; that is, we shall have

$$T(\tan\theta_2 - \tan\theta_1) = \alpha s + \beta x.$$

If s is small, this becomes

$$T\frac{d^2y}{dx^2} = \alpha\frac{ds}{dx} + \beta, \tag{1}$$

since

$$\tan\theta_2 - \tan\theta_1 = \frac{d}{dx}\left(\frac{dy}{dx}\right)dx.$$

But since

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + 1,$$

we may write

$$T\frac{d^2y}{dx^2} = \alpha\sqrt{1+\left(\frac{dy}{dx}\right)^2} + \beta.$$

If $p = dy/dx$, we have

$$\frac{dp}{a\sqrt{1+p^2}+b} = dx, \tag{2}$$

where $a = \alpha/T$ and $b = \beta/T$. The equation (2) is the differential equation of the curve. To integrate it, put $p = \tan\theta$; then

$$\frac{\sec^2\theta d\theta}{b + a\sec\theta} = dx, \quad \text{or} \quad \frac{d\theta}{\cos\theta(a + b\cos\theta)} = dx.$$

By the method of partial fractions, we may write

$$(3) \qquad \frac{d\theta}{\cos\theta} - \frac{b\,d\theta}{a+b\cos\theta} = a\,dx.$$

Integrating, we find

$$(A) \qquad \log(\sec\theta+\tan\theta) - \frac{b}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right] = ax,$$

the constant of integration being zero if we let $x=0$ when $\theta=0$. Dividing (2) by dy, we may write

$$(4) \qquad \frac{\frac{dp}{dy}}{a\sqrt{1+p^2}+b} = \frac{dx}{dy} = \frac{1}{p};\quad \frac{p\,dp}{a\sqrt{1+p^2}+b} = dy.$$

To integrate this put, as before $p=\tan\theta$. Then

$$\frac{\tan\theta\sec^2\theta\,d\theta}{a\sec\theta+b} = dy, \qquad \text{or} \qquad \frac{\sec\theta[d(\sec\theta)]}{a\sec\theta+b} = dy.$$

Integrating, we find

$$b+a\sec\theta-b\log(b+a\sec\theta) = a^2y+K.$$

If $y=y_0$ when $\theta=0$, $K=b+a-b\log(a+b)-a^2y_0$, and hence

$$(B) \qquad a(\sec\theta-1)+b\log\left[\frac{a+b}{b+a\sec\theta}\right] = a^2(y-y_0).$$

The equations (A) and (B) are the parametric equations of the curve, the parameter being θ, the angle of inclination.

3. *Alternative Forms. If $b<a$, equation* (A) *in its present form will involve square roots of negative quantities.* But in this case there is a form of (A) which is real, and which is therefore better suited for computation. This form will now be developed.

The second term of the left member of (A) is now

$$z = \frac{b}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right],$$

that is,

$$z = \frac{b}{i\sqrt{a^2-b^2}} \log\left[\frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right],$$

or

$$e^{\frac{iz\sqrt{a^2-b^2}}{b}} = \frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}$$

or

$$\cos\left(\frac{z\sqrt{a^2-b^2}}{b}\right)+i\sin\left(\frac{z\sqrt{a^2-b^2}}{b}\right) = \frac{b+a\cos\theta+i\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}.$$

Equating the imaginary parts,

$$\sin\left(\frac{z\sqrt{a^2-b^2}}{b}\right) = \frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta};$$

$$z = \frac{b}{\sqrt{a^2-b^2}} \sin^{-1}\left[\frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right]$$

so that (A) may be replaced by the equation

(C) $$\log(\sec\theta+\tan\theta) - \frac{b}{\sqrt{a^2-b^2}} \sin^{-1}\left[\frac{\sqrt{a^2-b^2}\sin\theta}{a+b\cos\theta}\right] = ax.$$

4. *Bridge Density Zero. If the linear density of the bridge becomes zero, the equations reduce to those of the common catenary.* Here $b = 0$, so (A) takes the simple form $\log(\sec\theta+\tan\theta) = ax$, since the second term reduces to zero. Hence we may write

$$e^{ax} = \sec\theta+\sqrt{\sec^2\theta-1}, \quad \text{or} \quad \sec\theta = \frac{1}{2}(e^{ax}+e^{-ax}).$$

The equation (B) becomes $\sec\theta-1 = a(y-y_0)$, since the argument of the logarithm never becomes infinite. Combining these results, we have

$$\frac{1}{2}(e^{ax}+e^{-ax}) = 1+a(y-y_0),$$

which is the equation of the common catenary.

5. *Cable Density Zero. When we put the linear density of the cable equal to zero, a becomes zero, and this makes* (A) *and* (B) *reduce to identities. We can, however, obtain the equations which should result.* If we divide (A) throughout by a and then put $a = 0$, the result is an indeterminate form, of the type zero divided by zero. This is evaluated by the rule of de L'Hospital:

$$\{b/(a^2-b^2)\}\Big\{[\sqrt{b^2-a^2}(b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta) - (a+b\cos\theta)(\sqrt{b^2-a^2}\cos\theta-a\sin\theta)] \div [(b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta)(a+b\cos\theta)] + \frac{a}{\sqrt{b^2-a^2}}\log\left[\frac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right]\Big\}.$$

When we now put $a = 0$, this reduces to $(1/b)\tan\theta$, and (A) takes the form

(A^1) $$\tan\theta = bx.$$

Similarly, when we divide (B) through by a^2 and take the quotient of the derivatives, we have

$$\frac{(\sec\theta-1)\left(1-\dfrac{b^2}{(a+b)(b+a\sec\theta)}\right)}{2a}\Bigg|_{a=0} = \frac{0}{0}.$$

Applying the rule a second time,

$$\frac{b^2}{2}(\sec\theta-1)\frac{b(1+\sec\theta)+2a\sec\theta}{(a+b)^2(b+a\sec\theta)^2}\Bigg|_{a=0} = \frac{1}{2b}\tan^2\theta,$$

so that (B) takes the form

(B^1) $$\frac{1}{2b}\tan^2\theta = y-y_0.$$

Combining (A^1) and (B^1), we may write

$$y-y_0 = \frac{b}{2}x^2,$$

which is the equation of a parabola.

THE UNIVERSITY OF CHICAGO

ON THE SOLUTION OF DIOPHANTINE EQUATIONS BY MEANS OF IDEALS*

BY G. E. WAHLIN

1. *Introduction.* In a previous article† I considered the application of the theory of ideals to diophantine analysis. At suggestions from Professors Dickson and Hedrick I have undertaken to make a study of the applications of the theory to problems of less general type than the one considered in that article. In this paper we shall consider diophantine equations in which one member is a binary quadratic form.

Since the publication of the previous article, I have noticed that, with a slight modification in the development, we could have obtained a theory for the solution of equations in which the left hand member is a more general decomposable form. This generalization would not involve any additional difficulty. In the following pages I have made use of this fact and have thus gained the advantage that, in the equations here considered, the only restriction imposed upon the binary quadratic form, used as the left hand member of the equation, is that its discriminant shall not be a perfect square.

In making the modification I have found it necessary to change the definition of a ring ideal from the one previously used in which we had the condition that a ring ideal is always relatively prime to the conductor. This restriction has been removed. Those ring ideals which are relatively prime to the conductor shall be called *regular* ring ideals. Since the properties of ring ideals as developed in Bachmann‡ apply only to regular ring ideals, it has

* Presented to the Society December 26, 1924.

† This BULLETIN, vol. 30, Nos. 3–4 (Mch.–Apr., 1924), p. 140. We shall refer to this article by the symbol (W).

‡ *Allgemeine Arithmetik der Zahlenkörper.*

been found necessary to insert the proof of a property of ring ideals in general. In speaking of equivalent ring ideals and classes of ring ideals, let it be understood that this refers only to regular ring ideals.

The theorems of §§ 2, 3, 4 of (W) refer only to regular ring ideals. Article 2 of the present paper is devoted to a general property of ring ideals. It is a modification of the development on pp. 152–3 of (W), and is used to simplify the following development.* As in (W) $\{\gamma\}$ is a principal field ideal and $[\gamma]$ is a principal ring ideal. Δ is used to denote the index and f the conductor of a ring.

2. *A General Property of Ring Ideals.* In this section I shall give a proof of the following theorem.

THEOREM I. *Let $\mathfrak{M}$ be an ideal of a ring R in an algebraic number field of degree n and γ an integer of $\mathfrak{M}$. Let $\{\gamma\} = T \cdot P_k$ where T is a field ideal relatively prime to $\mathfrak{M} \cdot f$, and P_k a field ideal whose prime ideal divisors are all divisors of $\mathfrak{M} \cdot f$. Then, if c is any rational integer divisible by $\mathfrak{M}$ and I a field ideal relatively prime to $\{c\Delta\}$ and belonging to the class reciprocal to that of T, we have $N(I) \cdot P_k / I = \{\gamma^{(k)}\}$ where $\gamma^{(k)}$ is an integer belonging to $\mathfrak{M}$.*

This theorem is obviously true if P_k is divisible by f, but requires a proof if this condition is not fulfilled. The proof does not, however, preclude this condition.

We shall denote by $\mathfrak{G}$ the unit ideal of the field. Since $\mathfrak{G}\mathfrak{M} \cdot f = \mathfrak{M}\mathfrak{G}f = \mathfrak{M} \cdot f$, $\mathfrak{M}f$ is a field ideal. As stated above let γ be any integer of $\mathfrak{M}$ and $\{\gamma\} = T \cdot P_k$ where T and P_k are field ideals satisfying the given condition. Since T is relatively prime to $\mathfrak{M}f$, it is relatively prime to f and hence there is a corresponding regular ring ideal $T^{(R)}$. From the ring class reciprocal to that of $T^{(R)}$ select an ideal $I^{(R)}$ such that the corresponding field ideal I is relatively prime to $\{c\Delta\}$. The possibility of such a choice

* By making use of the theorem of § 2 below, the development of (W) can be made without the introduction of the ideal I and the rational integer M_2.

of $I^{(R)}$ was shown in § 3, (W). $I^{(R)}$ is then relatively prime to $\mathfrak{M}f$ which is a divisor of $\{c\Delta\}$ and is also a ring ideal.

Denote the norm of I by M. Since M and $c\Delta$ are rational integers and I is relatively prime to $\{c\Delta\}$, M and $c\Delta$ are relatively prime, because if M and $c\Delta$ have a common factor let p be a rational prime which divides both M and $c\Delta$, some ideal prime factor of p must be a dividor of I and hence a common factor of I and $\{c\Delta\}$ contrary to our assumption regarding I. Hence $I_k = \{M\}/I$ is also relatively prime to $\{c\Delta\}$ and hence also to $\mathfrak{M}f$ and the corresponding ring ideal $I_k^{(R)}$ is regular.

Since

$$I_k \cdot I = \{M\}, \qquad I_k^{(R)} \cdot I^{(R)} = [M];$$

and since $I^{(R)}$ and $T^{(R)}$ belong to reciprocal classes in R, $I^{(R)} \cdot T^{(R)} = [\alpha]$, where $[\alpha]$ is regular. Multiplying both members of the last equation by $\mathfrak{G}$, we have $T \cdot I = \{\alpha\}$; and since T and I are both relatively prime to $\mathfrak{M}f$, we have

$$\mathfrak{M}f + \{\alpha\} = \{1\} = \mathfrak{G}.$$

Hence $\mathfrak{M}f + [\alpha]$ is the unit ideal of R. The ideals I_k and T belong to the same class, namely the one reciprocal to that of I; hence I_k and P_k belong to reciprocal classes and $I_k \cdot P_k = \{\gamma^{(k)}\}$.

We shall next see that with I_k as defined, the integer $\gamma^{(k)}$ may be chosen from $\mathfrak{M}$. $T \cdot P_k = \{\gamma\}$ where γ belongs to $\mathfrak{M}$. Since M is a rational integer $M\gamma$ belongs to $\mathfrak{M}$. If we then write

$$I_k P_k = \frac{\{M\}}{I} P_k = \frac{\{M\} T \cdot P_k}{IT} = \left\{\frac{M\gamma}{\alpha}\right\} = \{\gamma^{(k)}\},$$

we may choose $\gamma^{(k)} = M\gamma/a$; hence $\alpha\gamma^{(k)} = M\gamma$, and therefore $\alpha\gamma^{(k)}$ belongs to $\mathfrak{M}$. Therefore if a be any integer from $[\alpha]$, $a\gamma^{(k)}$ belongs to $\mathfrak{M}$. If b be any integer from $\mathfrak{M}f$ then $b\gamma^{(k)}$ belongs to $\mathfrak{M}f$ and hence to $\mathfrak{M}$. From this it follows that $(a+b)\gamma^{(k)}$ belongs to $\mathfrak{M}$. But we have seen that $\mathfrak{M}f + [\alpha] = [1]$; hence a and b may be so chosen that $a+b=1$ and therefore $\gamma^{(k)}$ belongs to $\mathfrak{M}$.

3. *Preliminary Computations for Rings in a Quadratic Field.* From now on we shall be concerned only with quadratic fields. The development of this article is that corresponding to § 2 of (W).

Let us consider a ring R in a quadratic number field $k(\sqrt{\mu})$. Let $(1, \varrho)$ be a fundamental system of R, so that all its numbers may be written in the form $x+y\varrho$, where x and y are rational integers. As in § 2 of (W), we can then write

$$\varrho^t = C_1^{(t)} + C_2^{(t)}\varrho,$$

and if ϱ is a root of the equation $x^2 - mx - n = 0$ we have

$$\varrho^2 = n + m\varrho. \tag{1}$$

We therefore see that

$$C_1^{(0)} = 1, \quad C_1^{(1)} = 0, \quad C_1^{(2)} = n,$$
$$C_2^{(0)} = 0, \quad C_2^{(1)} = 1, \quad C_2^{(2)} = m.$$

Multiplying both members of

$$\varrho^{t-1} = C_1^{(t-1)} + C_2^{(t-1)}\varrho$$

by ϱ and substituting for ϱ^2 its value from (1), we have

$$\varrho^t = C_2^{(t-1)} \cdot n + (C_1^{(t-1)} + m \cdot C_2^{(t-1)})\varrho,$$

whence we have the recursion formulas

$$C_1^{(t)} = n \cdot C_2^{(t-1)}, \qquad C_2^{(t)} = C_1^{(t-1)} + m \cdot C_2^{(t-1)}. \tag{2}$$

From this, if $\alpha^{(i)} = a_{1i} + a_{2i}\varrho$, we obtain

$$\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(k-1)} = B_0^{(k-1)} + B_1^{(k-1)}\varrho + \cdots + B_{k-1}^{(k-1)}\varrho^{k-1}$$
$$= A_1^{(k-1)} + A_2^{(k-1)}\varrho,$$

where

$$A_1^{(k-1)} = \sum_{i=0}^{k-1} C_1^{(i)} B_i; \qquad A_2^{(k-1)} = \sum_{i=0}^{k-1} C_2^{(i)} B_i.$$

We note also that

$$(A_1^{(k-1)} + A_2^{(k-1)}\varrho)(a_{1k}a + a_{2k}\varrho) = (a_{1k}A_1^{(k-1)} \cdot a + a_{2k}A_2^{(k-1)} \cdot n)$$
$$+ (a_{1k}A_2^{(k-1)} \cdot a + a_{2k}A_1^{(k-1)} + a_{2k}A_2^{(k-1)} \cdot m)\varrho.$$

4. *Quadratic Forms.* We shall next consider the binary quadratic form $ax^2+bxy+cy^2$ in which a, b, c are rational integers, $a>0$ and $b^2-4ac=g^2\mu$, where $\mu \neq 0$ or 1 and has no square factor. This form we shall denote by $Q(x, y)$, so that

$$a\cdot Q(x, y) = (ax+\varrho y)(ax+\varrho' y), \tag{3}$$

where $\varrho = (b+g\sqrt{\mu})/2$ and $\varrho' = (b-g\sqrt{\mu})/2$.

The numbers ϱ and ϱ' are the roots of the equation

$$x^2-bx+ac = 0. \tag{4}$$

Let us consider all numbers of the form $x+y\varrho$, where x and y are rational integers. Since the sum or difference of two such numbers is a number of the same form, they constitute a modul. Moreover it is easily shown that the product of two of these numbers is a number of the same form, and since for $x = 1$, $y = 0$ we get the number 1, we see that the modul is a ring R of which $(1, \varrho)$ is a fundamental system.

If we next consider all numbers of the form $ax+\varrho y$, they also constitute a modul $\mathfrak{M}$ in R and

$$\begin{aligned}(ax_1+\varrho y_1)(x_2+\varrho y_2) &= ax_1x_2+(ax_1y_2+x_2y_1)\varrho+y_1y_2\varrho^2\\ &= a(x_1x_2-cy_1y_2)+\varrho(ax_1y_2+x_2y_1+by_1y_2).\end{aligned}$$

Hence the product of any number of $\mathfrak{M}$ by a number of R is a number of $\mathfrak{M}$, and $\mathfrak{M}$ is therefore an ideal in R. The ideal $\mathfrak{M}$ has the fundamental system (a, ϱ).

Since

$$a = a\cdot 1+0\cdot\varrho, \qquad \varrho = 0\cdot 1+1\cdot\varrho,$$

$$\begin{vmatrix} a & 0 \\ 0 & 1 \end{vmatrix} = a$$

is the norm of the ideal $\mathfrak{M}$.

We shall next determine the index and conductor of R. Let $(1, \theta)$ be a fundamental system of $k(\sqrt{\mu})$. Then $\theta = \sqrt{\mu}$ if $\mu \equiv 2$ or 3 (mod 4) and $\theta = (1+\sqrt{\mu})/2$ if $\mu \equiv 1$ (mod 4). Let us consider the case when $\mu \equiv 2$ or 3 (mod 4). Then

$$(5) \qquad 1 = 1\cdot 1+0\cdot\theta, \qquad \varrho = \frac{b}{2}\cdot 1+\frac{g}{2}\cdot\theta.$$

Now $g^2\mu \equiv 0$ or $\mu \pmod 4$ and $b^2-4ac \equiv 0$ or $1 \pmod 4$. But $g^2\mu = b^2-4ac$, and hence if $\mu \not\equiv 1 \pmod 4$, $b^2-4ac \not\equiv 1 \pmod 4$, so that $g^2\mu \equiv 0 \pmod 4$ and g is even; hence also b is even and $b/2$ and $g/2$ are integers. From (5) we see that $\Delta = g/2$. In the case where $\mu \equiv 1 \pmod 4$, if $g^2 \equiv 0 \pmod 4$ also $b^2 \equiv 0 \pmod 4$, and if $b^2 \equiv 0 \pmod 4$ also $g^2 \equiv 0 \pmod 4$; hence b and g are simultaneously even or odd. We may then write

$$(6) \qquad 1 = 1\cdot 1+0\cdot\theta, \qquad \varrho = \frac{b-g}{2}+g\theta$$

and see that in this case $\Delta = g$. The conductor f of R is the totality of all integers of $k(\sqrt{\mu})$ such that the product of any integer of $k(\sqrt{\mu})$ by any integer of f belongs to R. Any integer of the field can be written in the form

$$(7) \qquad x+\theta y = x-\frac{b}{g}y+\frac{2y}{g}\varrho = \frac{gx-by}{g}+\frac{2y}{g}\varrho,$$

when $\mu \equiv 2$ or $3 \pmod 4$, and

$$(8) \qquad x+\theta y = x-\frac{b-g}{2g}y+\frac{y}{g}\varrho = \frac{2gx-by+gy}{2g}+\frac{y}{g}\varrho$$

when $\mu \equiv 1 \pmod 4$.

The question then remains to determine the nature of an integer α if $\alpha(x+y\theta) = \xi+\varrho\eta$, where ξ and η are rational integers. Since f is contained in R, α must be of the form $u+v\varrho$; hence when $\mu \equiv 2$ or $3 \pmod 4$, we have

$$\begin{aligned}\alpha(x+y\theta) &= u\cdot\frac{gx-by}{g}+\frac{2uy+gvx-bvy}{g}\varrho+\frac{2vy}{g}\varrho^2\\ &= \frac{u(gx-by)-2vyac}{g}+\frac{v(gx-by)+2uy+2bvy}{g}\varrho.\end{aligned}$$

It follows that u and v must be such that

$$\begin{aligned} u(gx-by)-2vyac &\equiv 0 \pmod g,\\ v(gx-by)+2uy+2bvy &\equiv 0 \pmod g;\end{aligned}$$

and since b and g are both even, this reduces to

$$(9)\quad \frac{b}{2}u+acvy \equiv 0, \qquad yu+\frac{b}{2}vy \equiv 0, \qquad (\text{mod } g/2).$$

Since y is arbitrary and hence in general not divisible by a factor of $g/2$, it may be divided out, and we have

$$\frac{b}{2}u+acv \equiv 0, \qquad u+\frac{b}{2}v \equiv 0, \qquad (\text{mod } g/2).$$

Since $b^2/4 \equiv ac \pmod{(g/2)^2}$, we see that the first congruence is obtained from the second by multiplying it by $b/2$ and hence any values of u and v which satisfy the second congruence also satisfy the first. We may therefore write $u = w\cdot g/2 - v\cdot b/2$, and hence

$$u+v\varrho = w\cdot\frac{g}{2}-v\cdot\frac{b}{2}+v\cdot\frac{b}{2}+v\cdot\frac{g}{2}\sqrt{\mu} = \frac{g}{2}(w+v\sqrt{\mu});$$

hence α must be a multiple of $g/2$. Moreover as is easily seen from (7), any integer of the field times $g/2$ is an integer of R. Therefore f is contained in $\{g/2\}$, and $\{g/2\}$ is contained in f, so that $f = \{g/2\}$.

In the case where $\mu \equiv 1 \pmod 4$, (8) shows that g, and hence also $\{g\}$, is contained in f. Again, let $u+v\varrho$ be any integer of f. Then

$$(u+v\varrho)(x+\theta y) = \frac{(2gx-by+gy)u-2acvy}{2g} + \frac{(2gx-by+gy)v+2uy+2bvy}{2g}\varrho,$$

and, as above, this leads to the congruences

$$(b-g)u+2acv \equiv 0, \quad 2u+(b+g)v \equiv 0, \quad (\text{mod } 2g);$$

or, since b and g are simultaneously odd or even,

$$(10)\quad \frac{b-g}{2}u+acv \equiv 0, \qquad u+\frac{b+g}{2}v \equiv 0, \quad (\text{mod } g).$$

Multiplying the second congruence by $(b-g)/2$, we have

$$\frac{b-g}{2}\cdot u+\frac{b^2-g^2}{4}v \equiv 0 \qquad (\text{mod } g).$$

But $b^2-4ac = g^2\mu$ and hence $b^2-g^2 = 4ac+g^2(\mu-1) \equiv 4ac \pmod{4g^2}$ since $\mu-1 \equiv 0 \pmod 4$; and therefore $(b^2-g^2)/4 \equiv ac \pmod g$. Again we see that any pair of values of u and v which satisfy the second congruence will also satisfy the first, and we may therefore write

$$u = wg - \frac{b+g}{2}\cdot v$$

and

$$u+v\varrho = wg - \frac{b+g}{2}\cdot v + \frac{b}{2}\cdot v + \frac{vg}{2}\sqrt{\mu}$$

$$= g\left(w + v\frac{1+\sqrt{\mu}}{2}\right) = g(w+v\theta).$$

Hence f is contained in $\{g\}$; and, as above, we conclude that $f = \{g\}$.

EXAMPLE. As an example of the foregoing, let us consider the form $Q(x, y) = 11x^2+5xy+2y^2$. The discriminant is -63; hence $g = 3$, $\mu = -7$, $\mu \equiv 1 \pmod 4$, $\Delta = 3$, $f = \{3\}$, $\varrho = (5+3\sqrt{-7})/2$, $R = (1, \varrho)$ and $\mathfrak{R} = (11, (5+3\sqrt{-7})/2)$. The number of classes in $k(\sqrt{-7})$ is one and in R it is four. If we denote the principal class of R by C_0 and the others by C_1, C_2, C_3, the following table gives a representative ideal for each class, the reciprocal class, and a binary quadratic form from the corresponding class of forms. To save space, the computations have been omitted.

Class	Representative ideal	Corresponding form	Reciprocal class	Representative ideal
C_0	$\left[1, \frac{5+3\sqrt{-7}}{2}\right]$	$x^2+5xy+22y^2$	C_0	$\left[1, \frac{5+3\sqrt{-7}}{2}\right]$
C_1	$\left[2, \frac{5+3\sqrt{-7}}{2}\right]$	$2x^2+5xy+11y^2$	C_2	$\left[2, \frac{5-3\sqrt{-7}}{2}\right]$
C_2	$\left[2, \frac{5-3\sqrt{-7}}{2}\right]$	$2x^2+5xy+11y^2$	C_1	$\left[2, \frac{5+3\sqrt{-7}}{2}\right]$
C_3	$\left[7, \frac{7+3\sqrt{-7}}{2}\right]$	$7x^2+7xy+4y^2$	C_3	$\left[7, \frac{7+3\sqrt{-7}}{2}\right]$

5. *Application to Diophantine Equations.* We shall now turn our attention to the study of the diophantine equation

$$Q(x, y) = u_1 u_2 \cdots u_{k-2}, \tag{11}$$

where it is required to determine the rational integral values of $x, y, u_1, u_2, \cdots u_{k-2}$ which satisfy the equation. Obviously the equation may be written in the form

$$N(ax+\varrho y) = au_1 u_2 \cdots u_{k-2}. \tag{12}$$

Following the procedure in (W), with the modification that $P_1, P_2, \cdots, P_{k-2}$ are constructed from prime ideals relatively prime to $\mathfrak{M}f$, we can show in the same manner that $u_i = \varepsilon_i \mu_i' \cdot \mu_i'' F_i(e_1^{(i)}, e_2^{(i)})$.

Before writing down the expressions for x and y, I shall make some observations regarding μ_i', and μ_i''. $|\mu_i' \cdots \mu_{k-2}'| = N(P_{k-1})$ where P_{k-1} is relatively prime to $\mathfrak{M}f$, but no μ_i' is itself the norm of an ideal. Hence each μ_i' contains only the first powers of such rational primes as are primes in $k(\sqrt{\mu})$ and every such prime must occur in an even number of the μ_i'. But since these primes are relatively prime to f and hence also to Δ, $x+\varrho y$ cannot contain such factors unless the x and y are both divisible by the prime, because with respect to such a prime as a modulus ϱ cannot satisfy a congruence of lower than the second degree. But such a solution can always be obtained from one in which the x and y do not have this common factor by multiplying it into x and y and any two of μ_i'. There is therefore no loss in assuming that $P_{k-1} = \{1\}$ and $I_{k-1} = \{1\}$; and hence $\alpha^{(k-1)} = +1$. This makes it possible to make $\mu_i' = +1$, $(i = 1, 2, \cdots, k-2)$, and in so doing we have in no way restricted the solution.

In accordance with what was stated above $P_1, P_2, \cdots, P_{k-2}$ are all relatively prime to $\mathfrak{M}f$ and P_k contains no ideal prime factors except such as are factors of $\mathfrak{M}f$. In writing the equation in the form (12) the factor a was introduced on the right hand side of the equation and now

$$N(P_k) = |a\mu_1'' \cdot \mu_2'' \cdots \mu_{k-2}''|, \qquad \text{where } a = N_R(\mathfrak{M}).$$

If we now use the method of (W) and the formulas in § 3 above, when we put $\gamma^{(k)} = a_{1k}a + a_{2k}\varrho$, we have the following formulas for the solutions of (11):

$$(13)\quad \begin{cases} x = \dfrac{1}{M}\left(a_{1k}A_1^{(k-2)} - a_{2k}A_2^{(k-2)} \cdot c\right), \\ y = \dfrac{1}{M}\left(a_{1k}A_2^{(k-2)} \cdot a + a_{2k}A_1^{(k-2)} + a_{2k}A_2^{(k-2)} \cdot b\right), \end{cases}$$

where $M = N(I_1 \cdot I_2 \cdots I_{k-2})$ as in (W);

$$(14)\qquad \mu_i = \varepsilon_i \mu_i'' F_i(e_1^{(i)}, e_2^{(i)}), \quad (i = 1, 2, \cdots, k-2),$$
$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k-2} = +1.$$

In these formulas the upper index on the A_i is $k-2$ in place of $k-1$ due to the fact that $\alpha^{(k-1)}$ was chosen equal to $+1$.

If $(\beta_1^{(i)}, \beta_2^{(i)})$ is a fundamental system of the ring ideal $I_i^{(E)}$ and

$$\beta_1^{(i)} = b_{11}^{(i)} + b_{12}^{(i)}\varrho, \qquad \beta_2^{(i)} = b_{21}^{(i)} + b_{22}^{(i)}\varrho,$$

we have, if $\alpha^{(i)} = a_{1i} + a_{2i}\varrho = e_1^{(i)}\beta_1 + e_2^{(i)}\beta_2$ is any integer of $I_i^{(E)}$,

$$(15)\quad a_{1i} = b_{11}^{(i)}e_1^{(i)} + b_{21}^{(i)}e_2^{(i)}, \qquad a_{2i} = b_{12}^{(i)}e_1^{(i)} + b_{22}^{(i)}e_2^{(i)}.$$

By means of (15) for $i = 1, 2, \cdots, k-2$ the expressions (13) for x and y may be obtained in terms of the parameters $e_1^{(i)}, e_2^{(i)}$, $i = 1, 2, \cdots, k-2$. The a_{1k} and a_{2k} are also parameters which enter into the expressions for the μ_i in the factors μ_i''.

The complete development of the foregoing part of this section is identical with that given in (W) except that here the $P_1, P_2, \cdots, P_{k-2}$ are all relatively prime to $\mathfrak{M}f$ and due to the development of § 2 above, the introduction of the ideal J and the rational integer M_2 is not needed. This is also true for (W) but I did not see it at the time.

The results can now be summarized in the following statement of a method for obtaining the solutions of (11). Select any integer γ from $\mathfrak{M} = (a, \varrho)$, and separate $\{\gamma\}$ into the product of two field ideals P_k and T, where T is relatively prime to $\mathfrak{M}f$ and P_k contains no prime factors

except such as are divisors of $\mathfrak{M}f$. Separate T into the product of $k-2$ field ideals $P_1, P_2, \cdots, P_{k-2}$ in any way and let $P_i^{(K)}$ be the ring ideal corresponding to P_i. From the ring class reciprocal to that of $P_i^{(K)}$ select an ideal $I_i^{(K)}$ which is such that the corresponding field ideal I_i is relatively prime to $\{a\Delta\}$ and put $I = I_1 \cdot I_2 \cdots I_{k-2}$. Let $N(I) = M$ and $I_k = \{M\}/I$ and $I_k \cdot P_k = \{\gamma^{(k)}\} = a_{1k} + a_{2k}\varrho$ by § 2. Let $F_i(x, y)$ be the binary quadratic form corresponding to the base $(\beta_1^{(i)}, \beta_2^{(i)})$ of $I_i^{(K)}$. Then the formulas (13) and (14) by means of relations (15) for $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_{k-2} = +1$ give solutions for (11) in terms of the parameters $e_1^{(i)}, e_2^{(i)}$ with the signs of the μ_i'' so chosen that the sign of their product is the same as that of $N(\gamma^{(k)})$.

By a variation in the separation of T into factors or by a variation in the choice of γ so that different classes are represented by $P_1^{(K)}, P_2^{(K)}, \cdots, P_{k-2}^{(K)}$ we can obtain all solutions of the equation. The values given to the parameters are rational integral and all solutions so obtained are rational integers.

EXAMPLE. Let us consider the equation

$$11x^2 + 5xy + 2y^2 = u_1 \cdot u_2 \cdot u_3.$$

Here $\mathfrak{M} = (11, (5+3\sqrt{-7})/2)$. Select, from $\mathfrak{M}$, $\gamma = 11 + 2(5+3\sqrt{-7})/2 = 16+3\sqrt{-7}$;

$$\{\gamma\} = \{2-\sqrt{-7}\}\{1+2\sqrt{-7}\}, \quad P_5 = \{2-\sqrt{-7}\}, \quad T = \{1+2\sqrt{-7}\}.$$

T is a prime ideal and hence its separation into $P_1 \cdot P_2 \cdot P_3$ gives $P_1 = \{1+2\sqrt{-7}\}$, $P_2 = \{1\}$, $P_3 = \{1\}$,

$$P_1^{(K)} = \left(29, \frac{13-3\sqrt{-7}}{2}\right), \quad P_2^{(K)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right), \quad P_3^{(K)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right).$$

$P_1^{(K)}$ belongs to the class C_1 (see table in § 4). $P_2^{(K)}$ and $P_3^{(K)}$ belong to class C_0.

$$I_1^{(K)} = \left(2, \frac{5-3\sqrt{-7}}{2}\right), \qquad F_1(x, y) = 2x^2+5xy+11y^2,$$

$$I_2^{(K)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right), \qquad F_2(x, y) = x^2+5xy+22y^2,$$

$$I_3^{(K)} = \left(1, \frac{5+3\sqrt{-7}}{2}\right), \qquad F_3(x, y) = x^2+5xy+22y^2,$$

$$I_1 = \left\{\frac{1+\sqrt{-7}}{2}\right\}, \qquad I_2 = \{1\}, \quad I_3 = \{1\},$$

$$I = I_1 \cdot I_2 \cdot I_3 = I_1, \qquad M = N(I) = 2,$$

$$I_5 = \{M\}/I = I_1' = \frac{1-\sqrt{-7}}{2},$$

$$I_5 \cdot P_5 = \left\{\frac{1-\sqrt{-7}}{2}\right\}\{2-\sqrt{-7}\} = \left\{-\frac{5+3\sqrt{-7}}{2}\right\} = \{\gamma^{(5)}\};$$

$\gamma^{(5)}$ is an integer of $\mathfrak{M}$. The sign may be neglected.

The linear forms used in computing (13) and (14), after application of (15), become

$$\alpha' = 2e_1^{(1)} + \frac{5-3\sqrt{-7}}{2} e_2^{(1)},$$

$$\alpha'' = e_1^{(2)} + \frac{5+3\sqrt{-6}}{2} e_2^{(2)},$$

$$\alpha''' = e_1^{(3)} + \frac{5+3\sqrt{-7}}{2} e_2^{(3)},$$

$$\alpha' \cdot \alpha'' \cdot \alpha''' \gamma^{(5)} = 11 \cdot 2x + \varrho \cdot 2y,$$

from which we have

$$\begin{aligned} x = {} & e_2^{(1)}e_1^{(2)}e_1^{(3)} - 2e_1^{(1)}e_2^{(2)}e_1^{(3)} - 2e_1^{(1)}e_1^{(2)}e_2^{(3)} \\ & - 10e_1^{(1)}e_2^{(2)}e_2^{(3)} - 22e_2^{(1)}e_2^{(2)}e_2^{(3)}, \\ y = {} & e_1^{(1)}e_1^{(2)}e_1^{(3)} + 5e_1^{(1)}e_2^{(2)}e_1^{(3)} + 11e_2^{(1)}e_2^{(2)}e_1^{(3)} + 5e_1^{(1)}e_1^{(2)}e_2^{(3)} \\ & + 11e_2^{(1)}e_1^{(2)}e_2^{(3)} + 3e_1^{(1)}e_2^{(2)}e_2^{(3)} + 55e_2^{(1)}e_2^{(2)}e_2^{(3)}. \end{aligned}$$

Since $N(P_5) = a = 11$, we find $\mu_1'' = \mu_2'' = \mu_3'' = 1$; and hence

$$\begin{aligned} u_1 &= \pm 2(e_1^{(1)})^2 + 5(e_1^{(1)}e_2^{(1)}) + 11(e_2^{(1)})^2, \\ u_2 &= \pm\ (e_1^{(2)})^2 + 5(e_1^{(2)}e_2^{(2)}) + 22(e_2^{(2)})^2, \\ u_3 &= \pm\ (e_1^{(3)})^2 + 5(e_1^{(3)}e_2^{(3)}) + 22(e_2^{(3)})^2. \end{aligned}$$

For $e_1^{(1)}=2, e_2^{(1)}=1, e_1^{(2)}=-1, e_2^{(2)}=2, e_1^{(3)}=-2, e_2^{(3)}=-1$, we have $x = 98$, $y = -181$, $u_1 = \varepsilon_1 29$, $u_2 = \varepsilon_2 79$, $u_3 = \varepsilon_3 36$, $\varepsilon_1\varepsilon_2\varepsilon_3 = +1$. For $e_1^{(1)}=1, e_2^{(1)}=1, e_1^{(2)}=-1, e_2^{(2)}=1, e_1^{(3)}=2, e_2^{(3)}=-1$, $x=24$, $y=-12$, $u_1=\varepsilon_1 18$, $u_2=\varepsilon_2 18$, $u_3=\varepsilon_3 16$, $\varepsilon_1\varepsilon_2\varepsilon_3=+1$.

For all $e_i^{(i)} = +1$, $x = -35$, $y = 91$, $u_1 - \varepsilon_1 18$, $u_2 = \varepsilon_2 28$, $u_3 = \varepsilon_3 28$, $\varepsilon_1\varepsilon_2\varepsilon_3 = +1$.

In order to obtain a different class of solutions, it would be necessary to make another choice of γ so that it can be separated into factors representing different classes from C_1, C_0, C_0.

6. *The Case* $k = 4$. When the diophantine equation to be solved is $Q(x, y) = u_1 \cdot u_2$, the same method as that used above will give a solution which may be easily written down

$$\begin{aligned}\alpha' &= e_1^{(1)}\beta_1^{(2)} + e_2^{(2)}\beta_2^{(2)},\\ \alpha'' &= e_1^{(2)}\beta_1^{(2)} + e_2^{(2)}\beta_2^{(2)},\\ \alpha^{(4)} &= a_{14}a + a_{24}\varrho,\end{aligned}$$

where $(\beta_1^{(1)}, \beta_2^{(1)})$ and $(\beta_1^{(2)}, \beta_2^{(2)})$ are bases of the two ideals I_1 and I_2. Let $F_1(x, y)$ and $F_2(x, y)$ be the corresponding binary quadratic forms. Then the solutions are given by the following relations:

$$(16)\qquad \begin{cases} x = \dfrac{a_{14}A_{12} - a_{24}A_{22}c}{M},\\ y = \dfrac{A_{14}A_2^{(2)} + a_{24}A_1^{(2)} + a_{24}A_2^{(2)}b}{M},\\ u_1 = \varepsilon_1\mu_1'' F_1(e_1^{(1)}, e_2^{(1)}),\\ u_2 = \varepsilon_2\mu_2'' F_2(e_1^{(2)}, e_2^{(2)}),\end{cases}$$

where

$$(17)\qquad A_1^{(2)} = \sum_{i=0}^{2} C_1^{(i)}B_i, \qquad A_2^{(2)} = \sum_{i=0}^{2} C_2^{(i)}B_i,$$

$$(18)\qquad \begin{cases} C_1^{(0)} = 1, & C_1^{(1)} = 0, & C_1^{(2)} = -ac,\\ C_2^{(1)} = 0, & C_2^{(1)} = 1, & C_2^{(2)} = b,\end{cases}$$

(19) $B_0 = a_{11}a_{12}, \quad B_1 = a_{11}a_{22}+a_{12}a_{21}, \quad B_2 = a_{21}a_{22},$

(20)
$$\begin{cases} a_{11} = b_{11}^{(1)}e_1^{(1)}+b_{21}^{(1)}e_2^{(1)}, \\ a_{21} = b_{12}^{(1)}e_1^{(1)}+b_{22}^{(2)}e_2^{(2)}, \\ a_{12} = b_{11}^{(2)}e_1^{(2)}+b_{21}^{(2)}e_2^{(2)}, \\ a_{22} = b_{12}^{(2)}e_1^{(2)}+b_{22}^{(2)}e_2^{(2)}, \end{cases}$$

where the $b_{ij}^{(k)}$ are constants depending on the ideals I_1 and I_2.

Let us now suppose that the ring R is the totality of all integers in $k(\sqrt{\mu})$. Then $(1, \varrho)$ is a fundamental system of the field; hence when $\mu \equiv 2$ or $3 \pmod 4$, $b = 0$ and when $\mu \equiv 1 \pmod 4$, $b = 1$. In either case $f = \{1\}$ and $\Delta = \pm 1$. No simplification would occur in the formulas (16) to (20) except that the b would be replaced by its value.

If, however, we make the additional restriction that $\mathfrak{M} = \mathfrak{G} = \{1\}$, we have $a = 1$, $P_k = \{1\}$; hence I_1 and I_2 belong to reciprocal classes, and in this case we may make $I_2 = I_1'$ and $F_1(x, y) = F_2(x, y)$. Moreover, since $P_k = \{1\}$ and $\mathfrak{M} = \{1\}$, we have $\mu_1'' = \mu_2'' = 1$, and the last two formulas of (16) become

$$u_1 = \varepsilon_1 F(e_1^{(1)}, e_2^{(1)}), \quad u_2 = \varepsilon_2 F(e_1^{(2)}, e_2^{(2)}), \qquad \varepsilon_1\varepsilon_2 = +1.$$

Since I_2 is the conjugate of I_1, we have $I_1I_2 = \{e\}$, where $N(I_1) = e$; hence $N(I) = e^2 = M$ and $\{M\}/I = I_4 = \{e\}$. Then $I_4P_4 = \{e\}\{1\} = \{e\}$. Hence $\gamma^{(k)} = e$ and $a_{14} = e_1$, $a_{24} = 0$. The first two of the formulas (16) now become

$$x = A_1^{(2)}/e = \frac{a_{11}a_{12} - a_{21}a_{22}c}{e},$$

$$y = A_2^{(2)}/e = \frac{a_{11}a_{22}+a_{21}a_{12}+a_{21}a_{22}b}{e}.$$

Since I_1 and I_2 are conjugate ideals and their norm is e, we can write the base of I_1 in the form $(e, f+\varrho)$ and that of I_2 in the form $(e, f+\varrho')$, where $f^2+bf+c \equiv 0 \pmod e$. Hence we have

$$b_{11}^{(1)} = e, \quad b_{21}^{(1)} = f, \quad b_{11}^{(2)} = e, \quad b_{12}^{(1)} = 0, \quad b_{22}^{(1)} = 1, \quad b_{12}^{(2)} = 0,$$

for all μ; and

$$b_{21}^{(2)} = f, \qquad b_{22}^{(2)} = -1,$$

when $\mu \equiv 2$ or 3 (mod 4); and

$$b_{21}^{(2)} = f+1, \quad b_{22}^{(2)} = -1,$$

when $\mu \equiv 1$ (mod 4).

Using these values for the $b_{ij}^{(k)}$ and combining the relations (16) to (20), we have, when $\mu \equiv 2$ or 3 (mod 4),

$$(21) \qquad \begin{cases} x = ee_1^{(1)}e_1^{(2)} + f(e_1^{(1)}e_2^{(2)} + e_2^{(1)}e_1^{(2)}) + ge_2^{(1)}e_2^{(2)}, \\ y = e_1^{(2)}e_2^{(1)} - e_1^{(1)}e_2^{(2)}, \end{cases}$$

where $g = \dfrac{f^2+c}{e}$. When $\mu \equiv 1$ (mod 4), we have

$$(22) \qquad \begin{cases} x = ee_1^{(1)}e_1^{(2)} + fe_2^{(1)}e_1^{(2)} + (f+1)e_1^{(1)}e_2^{(2)} + ge_2^{(1)}e_2^{(2)}, \\ y = e_1^{(2)}e_2^{(1)} - e_1^{(1)}e_2^{(2)}, \end{cases}$$

where $g = (f^2+f+c)/e$.

The binary quadratic form $F(x, y)$ may be chosen as the form corresponding to the base of I_1 as given above, and hence

$F(x, y) = ex^2 + 2fxy + gy^2$, when $\mu \equiv 2$ or 3 (mod 4);
$F(x, y) = ex^2 + (2f+1)xy + gy^2$, when $\mu \equiv 1$ (mod 4).

The formulas (16) as thus reduced are those given by Dickson* if we put

$$e_1^{(1)} = e, \quad e_2^{(1)} = g, \quad e_1^{(2)} = u, \quad e_2^{(2)} = -2.$$

A shorter proof of these formulas was published later by Dickson†. The proof as here given is by reduction from the general case by the following specializations: (1) $k = 4$; (2) $R = \mathfrak{G}$, the ring of all integers in $k(\sqrt{\mu})$; and (3) $\mathfrak{M} = \mathfrak{G}$.

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* This BULLETIN, vol. 27, No. 8 (May, 1921), p. 353.
† This BULLETIN, vol. 29, No. 10 (Dec., 1923), p. 464.

NÖRLUND ON CALCULUS OF DIFFERENCES

Vorlesungen über Differenzenrechnung. By Niels Erik Nörlund. Berlin, Springer, 1924. ix + 551 pages.

This is the first book to develop the theory of the difference calculus from the function-theoretic point of view and to include a significant part of the recent researches having to do with the analytic and asymptotic character of the solutions of linear difference equations. As such it is an important contribution to the mathematical literature and will render a service not procurable from any of its predecessors. The author presents a connected account of what appears to him to be the most important and the best developed domains of the difference calculus. The book is intended to give a preliminary view of the field and to facilitate the reading of the original memoirs. With this purpose in view the author sometimes omits proofs (too frequently, we think) leaving the reader to find them in the indicated memoirs. There is unfortunately little attempt to make clear the historical aspects of the subject, with the result that the casual reader of the book will get an erroneous impression of the actual historical development. It is the reviewer's opinion that this danger is not sufficiently safeguarded by the reference to Nörlund's *Neuere Untersuchungen über Differenzengleichungen* in the ENZYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, Band II C 7.

The most important earlier books on the difference calculus are those of Lacroix, Boole, Markoff, Pascal, Pincherle, and Wallenberg and Guldberg. They proceed principally from the algebraic standpoint (with a partial exception in the case of the last) and often employ symbolic methods. In Nörlund's book the point of view is function-theoretic throughout. This marks well the shift of interest on the part of those who have developed the difference calculus. The more formal aspects, elementary solutions, interpolations and such things engaged the attention of the founders, among whom were Newton, Taylor, Stirling, Laplace and Gauss. "Dann kam eine Zeit," as Nörlund says (in his introduction, page 2), "in der das Gebiet fast ganz brach lag und keine nennenswerten Fortschritte erzielt wurden. Heute jedoch zeigt sich wieder frisches Leben; unter neuen Gesichtspunkten und mit besseren Hilfsmitteln als früher hat man das Studium der zum Teil bis in die Anfänge der Differenzrechnung zurückreichenden Probleme wieder aufgenommen, und in den letzten Jahren ist eine ziemlich umfangreiche Literatur emporgewachsen, aus der als besonders wertvoll einige von Pincherle, Birkhoff, Carmichael, Galbrun und Perron herrührende Arbeiten zu nennen sind."

To this list of recent investigators the name of Nörlund himself is to be added and is to be given a place of large importance. We believe that the name of Poincaré should also be mentioned here as one who made an important contribution to the difference calculus in 1885 and thus gave the first intimation of the possible rejuvenescence of the theory of difference equations. Nörlund himself (p. 272) refers to this work of Poincaré as opening up for the first time a real theory of linear difference equations by means of a (now celebrated) theorem on the asymptotic behavior of the solutions of such equations.

The main part of Nörlund's text covers 453 pages. About one-third of the whole (making up the larger part of the first 200 pages) is given to the problem of finite integration and its congeners and generalizations. By a finite integral of a function $\varphi(x)$ we mean a function $f(x)$ satisfying the equation $[f(x+\omega)-f(x)]/\omega = \varphi(x)$. A related problem is that of determining a function $g(x)$ which satisfies the equation $[g(x+\omega)+g(x)]/2 = \varphi(x)$. In each of these equations ω is an arbitrary complex number. For many purposes ω may be given the value unity without loss of generality. But, for the theory of finite integration, as developed by Nörlund, it plays the role of an independent variable. The generalization of finite integration in the direction of what may be called repeated finite integration and a similar generalization of the related problem associated with the second of the foregoing equations are treated at length. As an example of the first we have the solution of the second order equation

$$\frac{f(x+\omega_1+\omega_2)-f(x+\omega_1)-f(x+\omega_2)+f(x)}{\omega_1\omega_2} = \varphi(x)$$

for $f(x)$ in terms of the given function $\varphi(x)$. Special cases of these problems are also treated at length; and, in particular, the determination of the finite integral of $\nu x^{\nu-1}$ when $\omega = 1$, together with the solution of related special problems. An elegant treatment is given of the theory of the usual and the higher Bernoulli and Euler polynomials.

The chief attention of the first seven chapters (except for a treatment of gamma functions and related functions in the twenty-one pages of Chapter V) is given to the various aspects of the problem of finite integration, or, as the author names it, that of finding the "sum" of a given function. The general finite integral of the given function $\varphi(x)$ involves an arbitrary periodic function of x of period ω. One principal purpose of Nörlund is to develop a direct algorithm by means of which finite integrals having characteristic properties can be selected from the totality of finite integrals. This purpose he achieves by means of appropriate definitions of summability applied to the direct formal sums

$$-\omega\sum_{s=0}^{\infty}\varphi(x+s\omega), \qquad \omega\sum_{s=1}^{\infty}\varphi(x-s\omega)$$

as formal finite integrals of the given function $\varphi(x)$. By this means he is able to set up appropriate definitions of finite integrals of such sort as to characterize them in a very satisfactory way and to render them determinate except for an additive constant. Thus in this book and in the relevant parts of his preceding memoirs the author brings the problem of finite integration to a stage of development comparable to that of integration in the infinitesimal calculus. Here his treatment has much of novelty and importance. One can only wish that he had more effectively brought out the relation of his own methods and results to those of other (and especially earlier) investigators who have had occasion to treat finite integration either for its own sake or as incidental to some larger problem in which it occurs.

Chapter eight (with 58 pages) is given to various types of interpolation series, with special attention to those of Stirling and Newton and to the character of functions defined as sums of such series.

An all-too-short development of the theory of factorial series is given in the ninth chapter, attention being confined to the most important results and especially to those which will be needed in later chapters.

Chapters ten to thirteen inclusive (pages 272–386) are devoted to the general theory of linear homogeneous difference equations. The first one opens with a general existence theorem developed by a method due to Nörlund. Then comes Ostrowski's recent proof of Hölder's theorem to the effect that the gamma function can not satisfy an algebraic differential equation, thus bringing out the fact that linear difference equations introduce us to a class of transcendental functions not brought to light by the study of differential equations. The generalizations or extensions of Hölder's theorem are not treated. A few elementary matters then follow and the chapter closes with a final section (of 13 pages) on Poincaré's theorem concerning the asymptotic character of solutions of difference equations.

The eleventh chapter (pages 314–352) is devoted to homogeneous linear difference equations with rational coefficients. It gives a rather extended development of a considerable part of the beautiful theory of these equations. The leading reference (page 314) is to Nörlund's 1915 paper in the ACTA MATHEMATICA and the methods employed in the chapter are due principally to him. In another and later part of the volume, namely on page 379 in the thirteenth chapter, is the statement that Birkhoff obtained (for these equations) many interesting and beautiful results "zu denen wir in Kapitel 11 unter anderen Gesichtspunkten gelangt sind." When it is remembered that Birkhoff's work was published in 1911, four years before that of Nörlund, it is seen to be desirable that Chapter eleven should contain a brief analysis of the relation between the work of the two authors with a clear indication of priority, especially if it is true (as I believe it to be) that the most

fundamental known properties of these equations are first developed in Birkhoff's memoir.

The twelfth chapter (pages 353–378) contains a development of the theory of those linear homogeneous difference equations whose coefficients can be expressed by means of factorial series. It contains an important contribution due to Nörlund himself. The method is one of the most pleasing yet known for treating the theory of linear difference equations; and I expect to see it developed by generalization so as to be still more effective. Factorial series afford one of the most important tools of analysis; and I believe that both they and certain of their generalizations have an importance beyond that which is generally recognized.

The treatment of linear homogeneous difference equations is brought to a close in the thirteenth chapter (pages 379–386) with a very short and imperfect account of the researches of Birkhoff in 1911. With 107 pages devoted to other contributions to the theory of homogeneous linear difference equations, it is not satisfying to see this important memoir passed over so briefly. If the material contained in these 107 pages requires that space for its proper presentation, then, it appears to me that another hundred pages is needed to present with proper distribution of emphasis the matter which is omitted. This would be taken mostly from the work of three out of the five recent investigators singled out by Nörlund for special mention in the passage which we have quoted from his introduction. In the absence of such a modified distribution of emphasis the exposition fails to give a correct impression of the historical order of development of the subject and of the relative importance of the results so far attained.

In the fourteenth chapter (pages 387–414) we have a treatment of the non-homogeneous linear difference equation. Section one contains an elementary account of Lagrange's method of variation of parameters. Equations with constant coefficients are treated in Section two. An account of the researches of E. Hilb in 1922 is given in the final Section three. It appears to me that the most important memoir on the solutions of non-homogeneous linear difference equations is that of K. P. Williams (Transactions of this Society, vol. 14 (1913), pp. 209–240). Nörlund nowhere in his book utilizes the results of this memoir by Williams, and there is indeed no reference to Williams in the index though his papers are listed in the bibliography. The failure to utilize Williams's researches leaves the chapter inadequate to represent the actual state of the theory of non-homogeneous equations.

The last chapter (pages 415–455) is devoted to reciprocal differences and continued fractions. It is followed by a few pages of tables, an extensive bibliography and an index. In the bibliography the author has included all works known to him and dealing with the general theory of difference equations. From the very extensive special litera-

tures of Bernoulli polynomials and gamma functions he has listed only the most important works. The bibliography is a very useful one. It is hardly to be expected that it should be complete. In fact I have found a considerable number of omissions by checking it against the partial bibliography which I have collected in an incidental way during the past fifteen years.

It is natural to expect that an exposition of a general subject should involve an important element depending on the personal interests of the author; and this is particularly true in the case of a book which is essentially the first in its field. But in the present book this element appears to me to have played too large a role in determining the distribution of emphasis and the selection of material. Much of the work in the first two hundred pages might well have been given with less fullness and the space so gained have been utilized in the presentation of some of the important matters which are omitted.

While this book will probably stand for some time as the best book in its field, and as such is therefore of great importance, it can not be regarded as having come near to being a definitive treatise on the difference calculus, even in its present state of development. Whatever one may think of the distribution of emphasis and selection of material in this volume there is still a definite need for another book with a quite different distribution and selection — one in which the personal equation of the author does not play so large a role.

R. D. CARMICHAEL

ENRIQUES ON ALGEBRAIC GEOMETRY

Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche. Vol. I and vol. II. By F. Enriques. Bologna, O. Chisini, 1915, 1918.

The reviewer, far from being a specialist in algebraic geometry, commences this short review with the misgiving that in the last two years of reading "at" the work of this famous author he has set his wisdom teeth into a sticky mouthful. It may however be said at the outset that the paucity, however unfair, of references to living workers in American universities,—a general reference to Osgood's *Funktionentheorie* and particular ones to Scott (C. A. Scott) and "Angas" ("Ch. Angas Scott"(!)),—indicates that the field is one in which few of us are expert, and therefore not only that the point of view of the reviewer will be that of most of his readers, but also that the treatise itself unbars a field which some of us might well explore. The reviewer tried the experiment of lecturing from it to capable, and patient, advanced students during the past academic year.

In the first place, although much elementary material is given, the prerequisites in the way of geometry and algebra are not meagre. Since many particular theorems of projective geometry are used and since the connection between algebra and geometry is obtained mainly by means of projective coordinates, which are independent of metrical axioms, the book should be preceded by a good study of projective geometry, such as Enriques's own, to which many references are given. The reviewer found to be even better adapted for this purpose the simpler logical structure of Veblen and Young, volume I and parts of volume II, of which he gave a three months analysis in the lectures just mentioned. On this basis, for example, the theory of linear systems and abstract elements in Book I is much simplified. As to prerequisites in algebra, one may assume that for advanced students such things as linear dependence and some ideas about substitutions and groups will be familiar; but can one say the same about topics like resultants and discriminants of functions of more than one variable and the Galois theory? The reader will save time by knowing something of these in advance. Perhaps finally, for good measure, he might have at hand Miss Scott's book as an introduction to the second volume and a general indication of the sort of problem to be considered.

The general plan of the treatise, in the author's words, is to investigate all those methods and concepts suggested originally by a study of plane algebraic curves, and to apply them in such a way as to constitute a qualitative account of algebraic equations and functions. So far, four books, in two volumes, have been published, viz., (I) a general introduction, (II) correspondences, (III) the theory of plane curves based on polars, (IV) the singularities of curves. A third volume is planned, in order to deal with the properties of curves invariant under birational transformations; perhaps in this the function-theoretic analysis will receive greater emphasis. Many methods are used for the analysis of the same problem, some for heuristic, some for rigoristic purposes and some for additional insight into the various ramifications of the problem. The various books stand as far as practicable on their own feet and may to that extent be read independently, and many of the "notes" or complements may be omitted in a first reading. Historical notes receive considerable emphasis throughout. What will much delight and instruct the reader is the frequent discussion of paradoxes and sophisms, which clarify the assumptions on which theorems rest.

Algebraic geometry is necessarily of the complex domain, as distinguished from that which seeks merely to illustrate the infinitesimal analysis of real functions, and equally naturally is, as far as achievable, a geometry *in extenso*. The methods which are developed may also be generalized and extended to give information about hyperspaces and algebraic varieties in an arbitrary number of dimensions.

Book I starts with an analysis of groups of points on a line and formations invariant under projective transformations, with special reference to groups of four points, involutions and the equations of the third and fourth degrees. Then the equation $f(x, y) = 0$ is analyzed, both as curve and as correspondence between projective forms of the first grade, and quadratic transformations are introduced to replace "infinitely near" for abstract geometry. The desirability of using the geometry of space of an arbitrary number of dimensions is seen not merely in the possibility of such generalizations as that of Plücker coordinates to hyperspaces immersed in an n-space, or linear systems as abstract geometries, but also in relation to the essential logical structure of the subject.

Consider in fact a question relating to the correspondence of algebraic varieties, in connection with Netto's theorem (Book I, Ch. III). Suppose that between the algebraic varieties V_n, V_m, themselves irreducible, there exists an irreducible correspondence such that to a generic point of V_n correspond ∞^r points of V_m ($r \geqq 0$), and a generic point of V_m corresponds to ∞^s points of V_n. We can imagine this correspondence established by a V_q in a V_{n+m},—in the same way as a correspondence between points x and points y on a line is defined by an $f(x, y) = 0$ in S_2,—and the V_q so formed will be irreducible. Netto's theorem, of which the author gives a special proof for algebraic varieties, then gives the result: $q = n + r = m + s$ or $n = m + s - r$. For simple applications of this theorem, for example to the finding of the dimension of the space of lines in S_3, where V_m is S_3 and the correspondence is of line to bundle, there is no difficulty; but in more complicated cases it would seem that, unless one took care actually to find the algebraic coordinates of the varieties involved, one's road would be tangled with questions of reducibility, and "sofismi" and "paradossi" would beset it.

Netto's theorem is thus made the basis for the meaning of "in general", e. g., the difference between "in general" for a correspondence $[m, n]$ and a curve of degree $m + n$, the counting of constants, the theory of dimensions, and the theory of compatibility of algebraic equations, leading to the principle of Plücker-Clebsch. The latter principle may be briefly stated in this form: if for generic values of parameters a set of algebraic equations is incompatible, then when it does for particular values become consistent it also becomes indeterminate; more exactly, if a system of $m = n - r$ equations in n unknowns ($r \geq 0$) is consistent and admits ∞^r, not ∞^{r+1}, solutions for some particular values of the parameters it is compatible (admitting ∞^r solutions) for generic values of the parameters.

In Book II the emphasis is on correspondences and the "principle of correspondence". The latter deals with the $m + n$ fixed points of an $[m, n]$ correspondence of points on a line. The relations between

these and the multiple points are given by Zeuthen's rule. The book contains a general theory of involutions of order higher than two, an elementary theory of plane curves and a chapter on the real representation of the imaginary and Riemann surfaces. Needless to say, the author's philosophical predilections make such general discussions extremely interesting.

The independence of the various books makes it possible, if one so desires, to read the next part, Book III, in large measure before Book II. In fact to many a reader the theory of plane curves based on polars, therein developed, will be the most familiar, although he will perhaps be a little startled by the discussion of one-sided (or odd) and two sided (or even) branches of curves, in the analysis of the real quartic in the projective plane. An algebraic curve without double points cannot have more than one odd branch, and if of even order cannot have any. The section on numerative geometry is extended to twisted curves and hyperspaces.

The last book starts again from elementary considerations and by means of Puiseux developments, quadratic transformations and differential methods, develops the theory of singularities of algebraic curves. The singularities of twisted curves and algebraic varieties are analyzed by means of projections; i. e.,— a curve in S_r is projected on a plane by means of the variable S_{r-2} which joins its generic point with a fixed S_{r-3}. An arbitrary plane curve, with arbitrary singularities, can arise in this way from a twisted curve in hyperspace which has no double points. Similarly, an arbitrary surface in S_3 is transformable into a surface F of a convenient hyperspace ($r \geqq 5$) without singularities, a fact by means of which is established the theorem that the surface can also by a birational transformation be transformed into another surface of S_3 with a double nodal curve having merely triple points (triple at the same time for the curve and for the surface).

For a person whose geometrical background is not more extensive than that of the reviewer some of the proofs may seem to be clouded by ambiguous wording or lack of sufficient detail, and some of the fundamentals, like the theory of quadratic transformations, not to be developed with sufficient broadness. The discussion of the relation of the equation of the fifth degree to the finite groups of projectivities is decidedly incomplete. Yet one has only to compare the book in quantity of material with Salmon, or in point of view and range of subject with almost any book in English on algebraic curves, to realize that it is the work of a rare scholar as well as a notable contributor to mathematics (and epistemology). Writers of minor and major theses are recommended to it.

G. C. EVANS

THREE REPRINTS FROM THE ENCYKLOPÄDIE

Neuere Untersuchungen über Funktionen reeller Veränderlichen, nach den Referaten von L. Zoretti, P. Montel, und M. Fréchet. By A. Rosenthal. Leipzig, Teubner, 1924. 351 pp.

Neuere Untersuchungen über trigonometrische Reihen. By E. Hilb and M. Riesz. Leipzig, Teubner, 1924. 40 pp.

Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. By L. Lichtenstein. Leipzig, Teubner, 1924. 58 pp.

The first article is a revision of the corresponding article in the French Encyclopedia, prepared under the direction of Professor Borel by Professors Zoretti, Montel and Fréchet. The original plan was to translate the French article as it stood, but a desire for greater completeness and accuracy led to an arrangement for its revision instead. As a result of the interruption in the work caused by the war, a ten year interval elapsed between the conclusion of this arrangement and the publication of the article. Hence the new material added in the revision took on the proportions of an entirely new article, the present version being about two and a half times the size of the original one. The additions are all indicated by the symbols $*\cdots*$, in accordance with the agreement made with Professor Borel and the usage in the French revision of articles appearing first in the German Encyclopedia.

The first part of the article, which forms the revision of the portion of the original article due to Zoretti, deals with the theory of point sets, and is based both on the literature prior to 1912 (the date of the French edition) and subsequent to that date. In the chapter entitled "Verallgemeinerungen" the author gives a brief account of work in General Analysis, to which he refers as "die weitgehenden Untersuchungen von E. H. Moore und seinen Schülern." He stresses particularly the very great generality of the results attained. A detailed account of these results is not given, since such an account would belong more properly in the article by Pincherle on functional operations and equations (ENCYCLOPÄDIE, II A 11), which was completed before the publication of any of the work in General Analysis.

The second part of the article forms the revision of the portion of the original version contributed by Montel, and deals with integration and differentiation. Among the noteworthy additions to be found in this part may be mentioned work on integration due to Pierpont, F. Riesz, Denjoy, Hellinger, and Perron, and the notion of the approximate derivative due to Khintchine and Denjoy.

The third part of the present article deals with sequences of functions and forms the revision of the first half of that portion of the original article which is due to Fréchet. The second half of Fréchet's contribution, dealing with trigonometric series, is omitted here, since that topic is completely covered by the article of Hilb and Riesz, reviewed below, and the previous article by Burkhardt (ENCYCLOPÄDIE, II A 12) on trigonometric series and integrals prior to 1850.

Among the important additions in this part to the topics treated in the original version may be mentioned the following: Osgood's fundamental results regarding the distribution of points in the neighborhood of which the convergence of a series is respectively uniform or non-uniform, together with generalizations of these results, the conception of relative uniform convergence due to E. H. Moore, the relation between the classification of Borel assemblages and Baire's classification of functions, the relations between measurable functions and Baire's classes, and the recent results of Blumberg relative to properties of entirely arbitrary functions.

The second article gives an excellent bird's-eye view of the extensive domain of mathematical theory with which it deals. It includes the more important results of all the modern researches in this field, beginning with Riemann's fundamental memoir, and gives occasional indications of methods of proof that have been particularly fruitful. In addition the numerous references to the literature found in the footnotes will serve to orient the reader in a more detailed study of any particular topic which he may select.

In one of the early sections the high lights in the development of the theory from the historical standpoint are indicated by the authors. These include Riemann's noteworthy contribution, in which the introduction of a general theory of integration greatly enlarged the scope of Fourier series, while at the same time the foundations were laid for a study of trigonometric series in general. Closely related to Riemann's work is Cantor's study of the uniqueness of the development, and Du Bois-Reymond's theorem that a convergent trigonometric series, representing a function with a Riemann integral, is necessarily a Fourier series, as well as the latter's examples of continuous functions with divergent Fourier series. Fejér's investigation of the summation of Fourier's series by Cesàro's means of the first order revealed the significance of the methods of divergent series in the field of trigonometric series. Finally Lebesgue's fundamental definition of the integral paved the way for developments that have added much to the unity and completeness of the whole theory.

After this brief sketch of the development of the theory, the logical rather than the historical order is used by the writers in their detailed accounts of methods and results. In dealing with Fourier series

these are grouped with respect to certain central topics, such as Fourier coefficients, general convergence theory, conjugate series, uniform and absolute convergence, special features of convergence and divergence, summation processes, Parseval's theorem, and the Riesz-Fischer theorem. In the presentation of the general theory of trigonometric series a summary of Riemann's work is followed by an account of the developments and generalizations of his ideas, due to various later writers. An appendix is devoted to the topics of multiple Fourier and trigonometric series and the approximate representation of continuous functions by means of trigonometric polynomials. The discussion of multiple series is extremely brief, it being pointed out by the authors that a general view of the present status of the theory may be gained by an examination of two papers dealing with this field, one by W. H. Young (PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, vol. 11 (1912)), and the other by H. Geiringer (MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 29 (1918)). The discussion of approximation theory is considerably more complete and includes the salient features of recent work in that field.

The third article connects up with the article by A. Sommerfeld (ENCYKLOPÄDIE, II A 7c) on boundary value problems in the theory of partial differential equations. The latter article was dated 1900 and gave an account of the developments in the field under consideration up to that date. Professor Lichtenstein's article deals with the additional developments since that time in the more restricted field indicated in his title.

As the author points out, the introduction of the methods of integral equations into the study of boundary value problems gave a very considerable impulse to the growth of the theory, and that portion of it dealing with linear differential equations of elliptic type may now be regarded as having reached a certain state of finality. A brief but comprehensive account of this portion of the theory, with complete references to the literature, is given in the present article. Even a casual examination of these references will readily reveal what an extensive portion of the recent developments in this field is due to Professor Lichtenstein himself.

The last two sections of the article are concerned with boundary value problems in the case of non-linear differential equations of elliptic type. Here, of course, the theory is much less developed, but many important additions have been made to it in the past quarter of a century. A systematic account of these researches, with references to the literature, is given in the sections mentioned.

C. N. MOORE

COURANT AND HILBERT ON MATHEMATICAL PHYSICS

Methoden der mathematischen Physik. Bd. I. By R. Courant and D. Hilbert. Berlin, Julius Springer, 1924. xiii + 450 pp.

The book under review is the twelfth volume of the series *Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen.* It is the second book of this series to be devoted to mathematical physics, it being preceded by volume IV, Madelung's *Die mathematischen Hilfsmittel des Physikers.** This earlier volume covers very extensive ground but, necessarily, in a rather cursory manner. The present volume, on the contrary, centers around one single physical problem, the oscillation problem, with its mathematical equivalents, the boundary value and expansion problems.

These are the main problems. Incidentally the reader will pick up a good deal about methods which are applicable to other problems of mathematical physics, but he will have to supply the applications himself. However, in these days of *Morbus relativitus* the information might be welcome that the word tensor appears on page 3 of the book and disappears on page 30, and it is not frequently used.

A few words regarding the joint authorship should be appropriate. The book is obviously and avowedly written by Courant. It is true that most of the subject matter originated directly or indirectly with Hilbert, whose spirit hovers over almost every page of the book. But the reader can easily verify, by looking up the several references, that a considerable portion of the book is based upon Courant's own investigations. This is especially the case with Chapter VI. Otherwise, the simple choice of methods, the fondness of heuristic considerations and a certain delicate touch of the pen, sometimes a bit vague but always elegant, betray the writer if nothing else does. All these qualities make the book easy and enjoyable reading.

We have already mentioned that the book deals with the oscillation problems of mathematical physics. This theory culminates in Chapters V and VI of the book, the former giving the equivalent boundary value and expansion problems, the latter the properties of the characteristic values and functions. The existence of the solutions is, occasionally postulated, many existence proofs being postponed to the second volume of the book which will appear later. The first four chapters lay a foundation for the theory; they deal with linear transformations and quadratic forms, expansions in terms of orthogonal functions, linear

* Review by B. O. Koopman, this BULLETIN, vol. 30, Nos. 5–6 (May–June, 1924), p. 272.

integral equations and the elements of the calculus of variations. In the seventh and last chapter some special functions defined by boundary value problems are dealt with in more detail.

The methods employed in the book are mostly very simple. Equicontinuity and Bessel's inequality are the only fairly novel tools which are used extensively. The integrals occurring are seldom interpreted in the sense of Lebesgue. Stieltjes and Hellinger integrals are completely avoided. In order to be able to place themselves on such an elementary basis the authors purposely restrict the discussion to functions which, together with the necessary number of derivatives, are continuous in adjacent intervals (stückweise stetig).

The leading idea in the greater part of the book is that the characteristic values and the characteristic functions* which belong to a given oscillation problem, satisfy certain *maximum-minimum conditions*. Thus the nth characteristic value is the upper limit for the minimum of a particular quadratic functional $F(\varphi)$ associated with the problem when the function φ is properly normalized and satisfies $(n-1)$ variable conditions of orthogonality. The function φ_n, supposed to exist, for which $\operatorname{Min} F(\varphi)$ reaches its upper limit, is the nth characteristic function; the corresponding $(n-1)$ orthogonal functions are the preceding characteristic functions. This *independent* definition of the nth characteristic value is due to Courant, though it seems to have been expressed for finite quadratic forms earlier by E. Fischer; the classical *recurrent* definition goes back to H. Weber. This simple principle is manipulated with great dexterity and yields surprisingly rich results.

The reader meets this guiding notion as early as on page 11 in the first chapter where the principal axes of a central hyper-quadric are found. It is clearly expressed for quadratic forms on page 17. The whole first chapter is essentially an algebraic analog from the theory of quadratic forms to the oscillation theory for continuous bodies developed later. It is the well known Hilbert theory oriented and adapted for a particular purpose. The second chapter opens with a general discussion of orthogonal functions, followed by a discourse on equicontinuous functions. The reader should notice the correction on page xiii referring to the discussion on page 40. In § 3 we meet the notions of *measure of independence* m and *asymptotic dimension number* r for a set of functions $f_i(x)$. The former is defined as the minimum of the quadratic form

$$\sum_{i,j=1}^{n} f_{ij} t_i t_j \quad \text{for} \quad \sum_{i=1}^{n} t_i^2 = 1, \quad \text{where} \quad f_{ij} = \int_a^b f_i f_j \, dx.$$

* Eigenwerte und Eigenfunktionen. The reader should notice that the authors use the term characteristic numbers (charakteristische Zahlen) for the reciprocals of the characteristic values.

If the functions $f_1, \cdots, f_n$ are linearly dependent, $m = 0$; if they are normalized and orthogonal, $m = 1$. The authors do not mention that this is the maximum value of m for normalized functions. Similarly the Gramian is discussed at various places, but the authors fail to notice that $\det(f_{ij}) \leqq f_{11} f_{22} \cdots f_{nn}$, where the equality sign holds if and only if the functions are orthogonal to each other.* The asymptotic dimension number is defined as the least integer r such that, if $s > r$, the measure of independence of the functions $f_{n_1}, f_{n_2}, \cdots, f_{n_s}$ converges to zero when $n_1, n_2, \cdots, n_s \to \infty$. The remainder of the second chapter is devoted to special orthogonal systems and to the approximation theorem of Weierstrass, Fejér's proof for trigonometric polynomials, and Landau's for ordinary ones.

In the third chapter we find an outline of the theory of integral equations. In addition to the classical theory of such equations, we find two new methods for the existence proofs. Both are based upon uniform approximation of the kernel by degenerate kernels, but the method of extracting a convergent sequence from the approximate solutions differ. One method employs the properties of equi-continuous functions, the other the asymptotic dimension number. For the symmetric case, the maximum-minimum properties of the characteristic values and functions play a fundamental role in the discussion. We miss a reference to Heywood and Fréchet's treatise in the bibliography appended to the third chapter.

The introduction is the most interesting part of Chapter IV, the elements of the calculus of variations. Here we are granted a foretaste of the direct methods of solving variational problems by means of minimal sequences which subject will be one of the main features of the second volume. In § 9 Hamilton's principle is introduced, with the aid of which the differential equations of vibrating masses are derived. The other parts of the chapter are more useful than exciting.

These differential equations form the main object of Chapter V. Here the boundary value problem is reduced to the solving of a symmetric integral equation by means of the corresponding Green's function. The existence of such a function is proved merely in the linear case; for two or more dimensions it is postulated and the proof will be given in the second volume. A solution of the expansion problem is given which is improved upon in the following chapter by reducing the restrictions. Various simple boundary value problems are treated explicitly.

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* For a proof see O. Dunkel, *Integral equalities with applications to the calculus of variations*, AMERICAN MATHEMATICAL MONTHLY, vol. 31 (1924), pp. 326–337. This interesting paper unfortunately does not give any references to the literature.

In Chapter VI we reach the climax. The maximum-minimum principle is given full play and is used for a very interesting qualitative discussion of the dependence of the characteristic values upon the data of the problem, namely the coefficients of the equation, the basic region and the conditions on the boundary. This discussion is facilitated by the following simple observation: The minimum of the functional $F(\varphi)$, when φ ranges over a certain field of functions, is not decreased by restricting the field, and not increased by enlarging the field. The same principle gives a simple determination of the asymptotic distribution of the characteristic values, the basic region under consideration being approximated by rectangles for which the characteristic values are known. An application to the black-body problem is given.

The seventh chapter is not so well done as the rest of the book. To be sure, there are brilliant points: the interpretation of Laplace integrals at the beginning of the chapter and the presentation of the saddle-point method at the end are gems. But the long and involved treatment of Bessel's functions seems scarcely justifiable except on the basis that contour-integration is a method of mathematical physics. It would have been simpler to start directly with the differential equation for $J_\lambda(x)/x^\lambda$ on page 399 and leave out most of the preceding discussion for which the reader could have been referred to the corresponding parts of Watson's standard treatise. A graver remark must be directed against the discussion of the zeros of Bessel's functions on pages 412–414. The formula (33) on page 413 becomes illusory for $\lambda<-1$, and the conclusions drawn in this case concerning the complex zeros of $J_\lambda(x)$ are erroneous. On page 419 Legendre's function of the second kind, $Q_\nu(x)$, is defined by an integral for $\Re(\nu)>-1$ and by the relation $Q_\nu(x)=Q_{-1-\nu}(x)$ for $\Re(\nu)<0$. This implies a contradiction in the common strip which could be avoided by making $\Re(\nu)=-\frac{1}{2}$ the dividing line between the two definitions. Why should the symbol $Q_\nu(x)$ represent different analytic functions of ν in different parts of the ν-plane? Why not use the original definition by a contour-integral which is valid for every value of ν, not a negative integer?

This finishes our survey of the different chapters. If our labor has not been in vain, it ought to be clear to the reader of this review that the book, in spite of its restricted scope, is rich in material and in points of view which are either novel or little known. The book — as most human work — is not perfect, but the imperfections are mostly on side-issues. It was obviously not meant as an opiate, but intended to stimulate interest, discussion and research in a field which still belongs to the richest in mathematical physics. We look forward to the appearance of Volume II with eager expectation.

EINAR HILLE

SHORTER NOTICES

Hydrodynamics. By Horace Lamb. Fifth edition. Cambridge, University Press, 1924. xvi + 687 pp.

This work reached its fifth edition in the forty-fifth year after its original publication in 1879, the successive editions having appeared at intervals of sixteen, eleven, ten, and eight years. In passing from the first edition to the second it was largely remodeled and extended, but since then there has been no change in general plan and arrangement. As it now appears it has again been carefully revised, several passages having been rewritten and some considerable additions having been made. It has, however, not undergone extensive enlargement. In his preface the author observes that "the work has less pretensions than ever to be regarded as a complete account of the science with which it deals," owing partly to the difficulty of doing justice to the growing literature. He adds: "Some memoirs deal chiefly with questions of mathematical method and so fall outside the scope of this book; others though physically important hardly admit of a condensed analysis; others, again, owing to the multiplicity of publications, may unfortunately have been overlooked. And there is, I am afraid, the inevitable personal equation of the author, which leads him to take a greater interest in some branches of the subject than in others." Though the book is extensive and important it is clear from these remarks that there are important aspects of the subject which it does not adequately treat — a fact which is stated without any implication of criticism, since it did not fall within the author's purpose to treat them and since the book probably serves better a large majority of its readers on account of the omission of certain subjects and the consequent fuller treatment of others.

The principal single increase in the scope of this work occurred in passing from the first to the second edition. The interpolations and additions incorporated in the third edition amounted to about one-fifth of the whole and thus required a renumbering of the sections. There was considerable increase in size in the fourth edition even though a few investigations of secondary interest were condensed or omitted. There are no omissions of consequence in passing from the fourth edition to the fifth; and there are several additions. Of the latter some are of the nature of interpolations (mostly short) inserted in the sections which are reproduced from the earlier edition, while the others treat new topics in the form of additional articles of which there are fourteen scattered through the volume. Throughout the whole work there

is repeated evidence of careful revision both in the addition of new matter and even in the smaller details of phraseology. The printing in the fifth edition, at least in the case of formulas, is a little more compact than in the fourth, and there is a considerable increase in the amount of matter in smaller type so that there is a reduction of twenty-one pages in size of the volume while there is an actual increase in matter.

As is to be expected in a fifth edition. the author adheres closely to the terminology which he had previously adopted. The most considerable change which came to the reviewer's attention is in the usage of the words "stream-line" and "stream-tube" as set forth in Sections 19 and 21; and even this is not of great importance.

In its new edition the book will continue to hold, and a little more effectively than ever before, the important place which it has now held for a generation.

R. D. CARMICHAEL

Geschichte der Elementarmathematik. By Johannes Tropfke. Bd. 7: *Stereometrie, Verzeichnisse.* Berlin and Leipzig, Walter de Gruyter & Co., 1924. iv + 128 pp.

The seventh and last volume of Tropfke's revised *History of Elementary Mathematics* maintains the high standard of the preceding volumes, and completes a work that is invaluable to the teacher of elementary mathematics, as well as to the student of the history of the subject. That the revision has been thoroughgoing is shown by the fact that the 35 pages of the first edition on solid geometry have been increased to 52 pages, while the number of references to the literature has been increased from 145 to 309. The very complete index covers 74 pages, and is arranged in two alphabets, under the headings "Names and Writings" and "Subject-matter".

Two items may be singled out for special mention, among the numerous improvements over the first edition. First, the additional information given concerning the history of technical terms is of great interest and value, especially for teachers, who will find such facts as are here to be found adding materially to the interest of many students in the subject-matter itself. Secondly, the history of the regular solids is enlarged and brought into line with the latest historical researches. The fact may be of general interest that the dodecahedron was known to the ancient Etruscans, who found it in natural form, in the crystals of iron pyrites that abound in northern Italy.

One error was noted. It is stated on page 32 that the theorem, "Sections of an oblique circular cone parallel to the base are circles", although known in antiquity, is first found explicitly proved in the *Book of the Three Brothers* in the 9th century. As a matter of fact, the proof occurs in Apollonius, along with that of the theorem that

the subcontrary sections are also circles.* On pages 40 and 41 occurs a passage where the revision was not edited so carefully as usual: Kepler is still, as in the first edition, credited with the cubature of the torus, whereas in the preceding paragraph (an addition in the new edition) this result is described as found in Heron of Alexandria. In this connection "the so-called theorem of Guldin" is mentioned. (See also p. 8.) Is it not time that Guldin's name was entirely dropped, and the theorem referred to exclusively as "the theorem of Pappus"? The reasons for doing this are clearly stated by Tropfke himself in the passage in question.

The index has evidently been very carefully compiled, which is of course an absolute necessity where the number of references is so great. The reviewer has checked a large number of these, and has noted the following corrections: p. 77. Kahun, I 134 should read, II 134; p. 101, Addition, II 65 should be I 65. Two other misprints should be mentioned: p. 51, l. 4, "Oktaeders" should read "Dodekaeders;" and in vol. IV, p. 139, l. 17, "4000 v. Chr." should read "2000 v. Chr."

Two topics that the reviewer believes should have been included have been entirely omitted: the *regula falsi*, and the "Russian peasant method of multiplication". Each is of historical interest and not without significance as furnishing supplementary material for class-room work or for outside reference, as in mathematics club programs.

We must end as we began, that this work is unique and of the highest excellence in its field, and a monument to the skill and industry of its author.

R. B. McCLENON

Binomial Factorizations, Giving Extensive Congruence Tables and Factorization Tables. Vols. II, III, and VI. By Lt. Col. Allan Cunningham. London, Francis Hodgson.

Of the seven volumes of this work the reviewer has in hand Volumes I, II, III, IV and VI. The content and general make-up of these tables has been considered in a previous review in the BULLETIN (vol. 31, pp. 180—181). By means of certain congruence-tables, giving the smallest roots, y, of the congruence

$$\frac{y^n \pm 1}{y \pm 1} \equiv 0 \pmod{p}$$

the author has built up factorization tables for numbers of certain special forms.

The value of such tables depends entirely upon their accuracy. An examination of certain of the tables, selected at random, leaves one in a state of doubt as to the value of the whole. Thus on page 193

* T. L. Heath, *Apollonius of Perga*, Cambridge, 1896, pp. 2, 3.

of Volume II are two little tables purporting to give the factors of what he calls "Trin-Aurifeuillian Quindecimans" and "Quint-Aurifeuillian Quindecimans", which, in plain English are factorizations of numbers of the forms $x^{15}+y^{15}$ after the algebraic divisors have been removed. Certain of the entries are left undecomposed, but at the head of the tables we are informed "All divisors less than 1,000 cast out" Nevertheless the following numbers on this page are left with a question mark:

$$17,780,401 = 151\times117,751$$
$$17,354,461 = 181\times95,881$$
$$15,679,621 = 211\times74,311.$$

The last number in the first table is 10,545,971 which is an error for $10,545,991 = 151\times211\times331$. Also the number 25,437,261 should be 125,437,261. Moreover the complete examination of all the numbers left doubtful on this page ought not to occupy a skilful computer many hours. One questions the usefulness or importance of publishing a table of factors of "Trinomial Dimorph Sextans", of eight entries, containing only five complete factorizations, and leaving five factors undecomposed two of which, $12,419,509 = 2029\times6121$, and $13,401,901 = 1297\times10,333$, would have yielded without much effort.

In spite of the fact that the author had access (p. 171, vol. I.) to the recently published factor-tables he leaves doubtful, in Volume I such numbers as

9,705,193, p. 171, vol. I,
9,670,849, p. 211, vol. I,
9,843,601, p. 217, vol. I,

and in Volume II we are given (p. 189) a question mark after 4,144,741.

These inaccuracies are, of course, not important in themselves, but show pretty clearly that these tables cannot be used in important work without careful checking.

D. N. LEHMER

Einführung in die Projective Geometrie Mehrdimensionaler Räume. By E. Bertini. Translated from the second Italian edition by A. Duschek. Wien, Seidel & Sohn, 1924. 480 pp.

In 1860 Cremona, then thirty years of age, became Professor of Higher Geometry at the ancient University of Bologna. About thirty years later Italy led the world in research in geometry. Due chiefly to Cremona's influence, the field of projective geometry in hyperspace was developed with great vigor. Today this subject can be said to be almost wholly Italian. Much good work on the subject has been done outside of Italy but nowhere else has there been such an army of the ablest mathematicians working at it. Professor Bertini of the University of Pisa has contributed his share in this work and it is,

therefore, fitting that the classical work on this subject should appear from his pen. Bertini has been lecturing for many years on n-dimensional projective geometry at Pisa and in 1899 a litographed edition of his lectures was published. This was revised and enlarged and finally appeared in print in 1907. The second edition appeared in 1923. It is this second edition which Dr. Duschek has translated. It has now been so many years since the appearance of the first edition that it will be unnecessary to discuss the contents. I will, therefore, only mention the added material which this second edition contains. The aim has been to bring the work up to date and new theorems are added in many places. One new chapter has been inserted, Chapter XII, Moduli, representation of a form by linear combinations of others.

Let $F_1 = 0$, $F_2 = 0, \cdots, F_h = 0$ be h algebraic hypersurfaces (of order $n_1, n_2, \cdots, n_h$) in n-space, which intersect in a variety of $n-h$ dimensions.

$F_1, F_2, \cdots, F_h$ are called *forms*, and $(F_1, F_2, \cdots, F_h)$ is called the *modulus*. A form F of order r is said to belong to the modulus if

$$F = A_1 F_1 + A_2 F_2 + \cdots + A_h F_h,$$

where the A_i are forms of order $r - n_i$. This is expressed by writing

$$F = 0, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

The fact that the difference of two forms $F - F'$ belongs to the modulus is expressed by

$$F' = F, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

A Schnitt of the modulus $(F_1, F_2, \cdots, F_h)$ is formed by connecting the coordinates by one or more linear relations.

With this introduction the problem is considered of finding the geometric conditions which the hypersurface $F = 0$ must satisfy if F can be written in the form

$$F = 0, \qquad \text{mod } (F_1, F_2, \cdots, F_h).$$

To Chapter XIV (XIII of the first edition) has been added a discussion of the Segre variety which is defined as follows: Let

$$\sum_{i=1}^{i=m} \lambda_i u_{ki} = 0, \qquad (k = 1, 2, \cdots, t),$$

where the λ's are parameters and u's linear functions, be the equations of t bundles of hyperplanes. The locus of points which will make the matrix of the u's of rank $m-h$ is the Segre variety and has many interesting properties.

The book has lost none of its attractiveness at the hands of the translator.

C. L. E. Moore

Elements of Analytic Geometry. Vol. I. By Neilos Sakellarios. Athens, E. and I. Blazoudake, 1924. vi + 288 pp. (In Greek.)

This elementary text for undergraduates is decidedly more reminiscent of Weber and Welistein than of Tanner and Allen. Admirable in its clarity, in the logical sequence of definitions and theorems, and in the frequent and abundant exercises provided for the student, there is throughout the book an exhaustive Teutonic precision and insistence on detail more properly characteristic of the treatise than of the elementary text, though it is only fair to add that this "classical" style of textbook writing has always been orthodox on the Continent. Examples of it are the rigorous distinction of line and sect, circumference and circle, geometric sum and difference from algebraic sum and difference; the systematic use of determinants, the inclusion of poles and polars for all conics, of the power of a point with respect to a circle and with respect to a sphere, and of the theorems of Menelaus and of Ceva.

From the outset, the analytic geometry of space is developed along with the analytic geometry of the plane. After a preface which traces the history of analytic geometry from Archimedes and Appolonius down to Plücker and Klein, the work is divided into five chapters. Chapters one and two, on completing the customary themes of coordinate axes, coordinates of a point, the various equations of the straight line, angles, and triangles, pass at once to the analogous discussion of the equations of the plane and of the line in space. Chapter three beginning with metrical properties in the plane, such as distance, perpendicularity, angle bisection, area of the triangle, leads likewise to the treatment of metrical properties in space. Similarly, chapter four makes use of the analysis of the circle to introduce the analysis of the sphere. The last chapter treats fully the ellipse, the hyperbola, and the parabola, in the order named, and concludes with a short comparison of curves of the second degree, algebraically, as loci of the general equation of the second degree, and geometrically, as sections of the cone. The analogous solid-analytic material of the last chapter, the treatment of quadric surfaces, is evidently reserved for the projected second volume, as is no doubt also the consideration of higher plane curves. The table of formulas at the end, as well as the printing of the statement of each theorem in heavy type, deserves praise. There are few typographical errors.

Any good book on geometry in the language of the race which gave geometry to the world should be of interest to mathematicians conversant with that language, especially so finely wrought a text as this is, in spite of the fact that the method of approach is not the ordinary one.

G. Morgenstern

Complementi di Geometria Descrittiva—visibilita—ombre—chiaroscuro—prospettiva lineare. By Gino Loria. Milano, Ulrico Hoepli, 1924. xii + 191.

In his preface the author states that this will probably be the concluding volume of his series of books on descriptive geometry published by this firm. (The earlier books were (1) *Metodi della Geometria Descrittiva,* 1st ed., 1909, 2d ed., 1919; (2) *Poliedri, Curve e Superfici, secondo i Metodi della Geometria Descrittiva,* 1912; (3) *Storia della Geometria Descrittiva della Origini Sino ai Giorni Nostri.* The two latter are reviewed in this BULLETIN, vol. 20, p. 415, and vol. 28, p. 414.)

The eight chapters are grouped into three "books". The first, whose title is the same as the principal title of the volume, has chapters on visibility, on surfaces of the second order, and on ruled surfaces. Book II (shadows and chiaroscuro) has chapters entitled shadows cast by points, lines, and plane polygons; shadows on surfaces; ideas concerning chiaroscuro; and the illumination of surfaces. These two books are of the same length; but book III (linear perspective) is much briefer (18 pages) and attempts nothing beyond "definitions, constructions, and formulas".

In writing this brief introduction to the applications of descriptive geometry, the author had in mind the student with a good geometrical background rather than the draftsman who wants merely a set of "working rules" for making perspective drawings and putting in shades and shadows.

E. B. COWLEY

Traité du Calcul des Probabilités et de ses Applications. By E. Borel, Tome III, *Les Applications de la Théorie des Probabilités aux Sciences Economiques et Biologiques.* Fascicule I. *Assurances sur la Vie. Calcul des Primes,* by Henri Galbrun. Paris, Gauthier-Villars, 1924. 311 pp.

This volume, the first part of a work on the mathematical side of life insurance, is just the sort of a book that we, here in America, would expect to be written by a professor of mathematics who was interested in life insurance. The best known writers in English on actuarial mathematics are actuaries who write for actuaries and who ever keep the practical applications in view. For them mathematics is simply a means to an end. The author is a French actuary and his book well illustrates the somewhat different view points of practical actuaries in this country and in France. Some allowance must, however, be taken for the fact that this volume is a part of a monumental work on the theory of probability and its applications.

Galbrun starts with the law of large numbers and proceeds in a straightforward way through the topics Bernoulli's theorem, Tchebycheff's theorem, interest, tables of mortality, Makeham's law, interpolation,

summation, approximate integration, pure endowments, life annuities, premiums on various kinds of life insurance, joint life and survivorship problems, loading of premiums. This is all done in much the usual way, but more attention is paid to the elegance of the mathematical arguments and the calculus is used more freely than in many books on actuarial mathematics. A feature of the book is the discussion of the errors involved in the calculation of endowments, annuities, policy premiums, and so on. Along with average rates he computes a sort of safe rate which will cover all but one percent of the possible cases which may arise. In places, the methods used in the book will lead to much more arithmetic drudgery then is necessary, but on the whole it is a thoroughly readable and scholarly book on that part of actuarial mathematics which it covers.

A. R. CRATHORNE

Leitfaden zum Graphischen Rechnen. By Rudolf Mehmke. 2d edition. Leipzig and Vienna, Franz Deuticke, 1924. viii + 183 pp.

Whether one does or does not believe in the practical value of graphic methods one must take seriously this guide to the subject. It is a mine of ingenious devices. Here are not only the usual graphic constructions for expressions of the form $\sum ab/c$ together with solutions of linear equations in two, three, four and more unknowns, with logarithmic scales for the more complicated functions, but exhaustive discussion of methods for integrating differential equations and the determination of moments of various sorts. In connection with logarithmic methods is a description of a "logarithmic compass" devised by Professor E. A. Brauer which has three points, the distance between two of which is a function of the distance between any other pair. Interesting use is made of this device.

As to the practical use of graphical methods one may be permitted to have serious doubts. Professor E. T. Whittaker, in his preface to his *Calculus of Observations*, remarks: "When the Edinburgh Laboratory was established in 1913 a trial was made, as far as possible, of every method which had been proposed for the solution of the problems under consideration, and many of these methods were graphic. During the ten years which have elapsed since then, the graphic methods have almost all been abandoned, as their inferiority has become evident, and at the present time the work of the Laboratory is almost exclusively arithmetic. A rough sketch on squared paper is often useful, but (except in descriptive geometry) graphic work performed carefully with instruments on a drawing-board is generally less rapid and less accurate than the arithmetic solution of the same problem."

With this judgement those who have learned to use a computing machine will be apt to agree. However, the subject of graphic computation is in itself a very interesting one, and there is also the chance that certain problems in geometry and mechanics which are too complicated for arithmetic analysis may yield to some such methods as we have here. It is important that every method be as fully developed as possible whether it may compare favorably or unfavorably with other methods. This has been done for the method of graphic computation in this book with scholarly thoroughness.

D. N. LEHMER

From Determinant to Tensor. By W. F. Sheppard. New York, Oxford University Press, American Branch, 1923. 127 pp.

This is an excellent little book the aim of which is to familiarize the student with tensors and to give an idea of their applications. It is, perhaps, not sufficiently realized that tensors are not merely a part of "relativity", the fact being that they permeate almost all mathematics.

Professor Sheppard gives illustrations of the applicability of tensor algebra to statistical theory and this chapter of his book is the most novel. The fact that tensor theory would be useful in this theory is à priori evident from the prominent position given in statistics to a quadratic form.

What caught the reviewer's attention first, on reading the book, was that Professor Sheppard had a happy thought when he tried to introduce the beginner to tensors by starting with determinants. The connection between alternating tensors, which are those that occur naturally in the study of geometrical figures and their attached integrals, and determinants is a deep-lying one and it is in some respects a good plan to reverse the process and obtain the principal results on determinants from very elementary theorems of tensor algebra. A very good account of determinants and their uses is given in the first few chapters of the book.

We wish to recommend this book heartily, but we must refer briefly to a point in it which we regard as unfortunate. Reduced to ordinary vector symbolism, the writer says that we can write the scalar product idea in the form $m/\overline{A} = \overline{B}$ this being understood as equivalent to $m = (\overline{A} \cdot \overline{B})$ where m is a scalar quantity and $\overline{A}$ and $\overline{B}$ are two vectors. If we also have $n/\overline{C} = \overline{B}$ the author equates $m/\overline{A}$ to $n/\overline{C}$ and says that $m/\overline{A} = n/\overline{C}$ is a way of stating that m is the same linear function of $\overline{A}$ as n is of $\overline{C}$ (p. 75). But $\overline{B}$ is not determined by $m/\overline{A}$, as we can write $m/\overline{A} = \overline{B} + \lambda\overline{X}$ where $\overline{X}$ is any vector perpendicular to $\overline{A}$. We do not, therefore, follow the author in his discussion of the

"direct proportion of simple sets" (vectors) but this disagreement is after all a small one, being largely a question of terminology.

The beginner will find the book a valuable introduction and the expert an interesting review with a refreshing novelty of presentation.

F. D. MURNAGHAN

The Mathematical Groundwork of Economics: An Introductory Treatise. By A. L. Bowley. Oxford, Oxford University Press, 1924. vi + 98 pp. $2.35.

Professor Bowley's prefatory statement of the purpose and scope of his book is an adequate description of his accomplishment as well as of his aim. He says "Though the simple application of mathematics made by competent writers and lecturers can be appreciated by any intelligent readers and students, the more complicated analyses are only within the power of those who have mathematical aptitude, and it is for them that this book is arranged. . . . . I have attempted to reduce to a uniform notation, and to present as a properly related whole, the main part of the mathematical methods used by Cournot, Jevons, Pareto, Edgeworth, Marshall, Pigou, and Johnson, so far as these are applied to the fundamental equations of exchange and to the elementary study of taxation. . . . I have not intended to advance any new theorems in economics, nor do I claim any originality in mathematical results."

The author points out that the science of wealth and welfare has two aspects, one subjective, the other objective. On the subjective side, satisfactions and desires are not arithmetically measurable, but, on the objective side, goods and services are measurable in terms of quantity and price. "At first sight," he continues, "it might appear that mathematical reasoning was confined to the objective aspect, but this is not the case." Mathematical methods may be applied to the subjective as well as the objective aspect of economics, for we can detect "equality and inequality, relationship, continuity, variation, and other properties" of satisfactions and desires. The mathematical treatment of economic concepts, therefore, embraces both incommensurable and measurable entities.

The chief contribution of the author in this book is to make available a uniform, orderly, and consistent treatment of the concepts of economics which lend themselves to mathematical analysis. The book will prove to be a valuable reference work for the economist with mathematical training. But it will have still greater value and influence if its applications of mathematics to economics are used as illustrative material by teachers of mathematics in their courses in analytic geometry and calculus.

W. M. PERSONS

NOTES

Mr. G. Y. Rainich, of Johns Hopkins University, and Professor J. F. Ritt, of Columbia University, have been appointed associate editors of the TRANSACTIONS OF THIS SOCIETY.

The following university courses for the academic year 1925–1926 are announced:

COLUMBIA UNIVERSITY.—By Professor T. S. Fiske: Fundamental concepts of mathematics; Differential equations.—By Professor F. N. Cole: Theory of groups.—By Professor C. J. Keyser: Modern theories in geometry; Introduction to mathematical philosophy (first semester).—By Professor D. E. Smith: History of mathematics, first and second courses (first semester); History of mathematics, advanced course (first semester); Seminar in the history of mathematics (first semester).—By Professor Edward Kasner: Seminar in differential geometry.—By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Elliptic functions (first semester); Advanced course in the theory of functions of a complex variable (second semester).—By Professor G. A. Pfeiffer: Analysis situs (second semester).—By Dr. M. H. Stone: Fourier series and related topics.

HARVARD UNIVERSITY.—By Professor W. F. Osgood: Advanced calculus; Functions of real variables; Linear differential equations of the second order, complex variables.—By Professor J. L. Coolidge: Subject matter of elementary mathematics; Probability; Algebraic plane curves.—By Professor E. V. Huntington: Fundamental concepts of mathematics.—By Professor O. D. Kellogg: Elementary theory of differential equations; Theory of potential functions; Dynamics (second course).—By Professor G. D. Birkhoff: Space, time, and relativity.—By Professor W. C. Graustein: Introduction to modern geometry; Projective geometry; Non-euclidean geometry.—By Dr. H. W. Brinkmann: Theory of functions; Partial differential equations of mathematical physics.—By Professor E. T. Bell (University of Washington): Theory of numbers.—By Dr. L. M. Graves (National Research Fellow): Calculus of variations.—By Mr. B. O. Koopman: Analytical theory of heat, problems in elastic vibrations; Modern methods in dynamics. Professor Bell and Dr. Brinkmann will conduct a fortnightly seminar in analysis and the theory of numbers. Courses of research are also offered by Professor Osgood in analysis, by Professor Coolidge in geometry, by Professor Kellogg in potential theory, by Professor Birkhoff in the theory of differential equations, by Professor Graustein in geometry, and by Dr. Brinkmann in the theory of groups.

From forty to fifty fellowships for advanced study abroad will be awarded annually by the John Simon Guggenheim Memorial Foundation, beginning with the academic year 1926–27. These fellowships are open to men and women; research or other productive work may be pursued in any academic subject or in the arts. The annual stipend is about $2500. Applications should be sent to the office of the Foundation, 2300 Pershing Square Building, New York City.

Applications for American Field Service Fellowships for study in French universities for the academic year 1926–27 should be sent to Stephen P. Duggan, 522 Fifth avenue, New York City, not later than December 15, 1925. Mr. W. E. Byrne, of Rensselaer Polytechnic Institute, has been appointed fellow in mathematics for 1925–26.

The Belgian Academy announces the following subjects for prizes in mathematics for the year 1926: *A contribution to the infinitesimal geometry of curved surfaces*; *A contribution to the theory of the absorption of light in interstellar space.*

In 1914 a committee was formed under the auspices of the Paris Academy of Sciences, with Professor Paul Appell as chairman, to found a prize and medal in honor of Henri Poincaré, for the encouragement of research in the fields of mathematical analysis, celestial mechanics, mathematical physics, and the philosophy of sciences. The work of this committee, which was interrupted by the war, has now been taken up again, and it is hoped to begin the annual awards shortly.

Cambridge University has awarded a Smith's prize to Mr. T. G. Room, of St. John's College, for an essay entitled *Varieties generated by collinear stars in hyperspace.* The Adams prize for an essay on *The physical state of matter at high temperatures* (see this BULLETIN, vol. 29, p. 235) has been awarded to Mr. R. H. Fowler, of Trinity College. The subject announced for the Adams prize for 1925–26 is *The constitution of the interior of the earth and the propagation of waves through the interior and over the surface of the earth.*

The Barnard medal has been awarded by Columbia University to Professor Neils Bohr, of the University of Copenhagen, in recognition of his work on atomic structure.

On the occasion of his recent visit to Buenos Aires, Professor Albert Einstein was elected an honorary member of the Argentine Academy of Science.

Professor Willy Wien, of Würzburg, has been elected an honorary fellow of the Physical Society of London.

Oxford University has conferred an honorary doctorate on Professor Paul Appell.

Professors Paul Appell and Emile Picard have been elected honorary members of the Russian Academy of Sciences.

Professor L. Lecornu has been promoted to be a commander of the Légion d'Honneur.

Eugène Fichot, director of the French naval hydrographic service, and known for his work in tidal theory, has been elected a member of the Paris Academy of Sciences in the section of geography and navigation.

The following mathematicians have been recommended by the president and council of the Royal Society of London for election as fellows: R. H. Fowler, Harold Jeffreys, Joseph Proudman.

The University of Edinburgh has conferred an honorary degree on Professor A. S. Eddington.

Dr. J. H. Jeans has been elected president of the Royal Astronomical Society.

Professor Richard Gans, of the University of La Plata, has been appointed professor of physics at the University of Königsberg.

Professor Hans Rademacher, of the University of Hamburg, has been appointed professor of mathematics at the University of Breslau.

Professor Lothar Schrutka, of the Brünn Technical School, has been appointed professor of mathematics at the Technical School at Vienna.

Professor G. Togliatti, of Turin, has been appointed professor of applied mathematics at the University of Zürich.

At the University of Paris, Professor Paul Montel has been transferred to the professorship of rational mechanics; Professor Gaston Julia succeeds him in the chair of general mathematics. J. Chazy has been appointed chargé de cours.

A readership in mathematical physics has been created recently at the University of Bristol, from the Wills bequest; Dr. J. E. Jones has been appointed the first reader.

Mr. W. L. Fellar, of the University of Edinburgh, has been appointed to a tutorial fellowship at Hertford College, Oxford.

The following 34 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1924; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case.

R. W. Babcock, Wisconsin, October, *On thermal convection.*

Herman Betz, Yale, June, *Surface transformations applied to dynamical systems with two degrees of freedom.*

A. D. Campbell, Cornell, September, philosophy, *Linear systems of conics in the Galois field.*

Elizabeth Carlson, Minnesota, June, physics, *On the convergence of certain methods of closest approximation.*

G. H. Collignon, Johns Hopkins, June, geophysics, *Problems of flow in connection with mapping of spherical polygons.*

Julia T. Colpitts, Cornell, June, philosophy, *On a certain class of entire functions.*

Julia Dale, Cornell, June, physics, *Some properties of the exponential mean.*

Marguerite D. Darkow, Chicago, August, *Arithmetics of certain algebras of generalized quaternions.*

W. W. Elliott, Cornell, June, physics, *Generalized Green's functions for compatible differential systems.*

Bess M. Eversull, Cincinnati, June, physics, *The summability of the triple Fourier's series at points of discontinuity of the function developed.*

F. J. Gerst, Johns Hopkins, June, geophysics, *Image points and Riemann's theorem.*

Cornelius Gouwens, Chicago, August, *Invariants of the linear group modulo* $p_1^{\lambda_1} \cdots p_n^{\lambda_n}$.

L. M. Graves, Chicago, August, *The derivative as independent function in the calculus of variations.*

C. F. Gummer, Chicago, August, *The relative distribution of the real roots of a system of polynomials.*

J. W. Hedley, Chicago, August, *Ruled surfaces whose flecnode curves belong to linear complexes.*

P. E. Hemke, Johns Hopkins, June, physics, *A transformation involving ζ-functions with an aerodynamical application.*

J. L. Holley, Harvard, *Linear spaces and their fixed points.*

Harold Hotelling, Princeton, June, *Three-dimensional manifolds of states of motion.*

Jewell C. Hughes, Chicago, August, *A problem in the calculus of variations in which one end-point is variable on a one-parameter family of curves.*

Mildred Hunt, Chicago, June, *The arithmetics of certain linear algebras.*

M. H. Ingraham, Chicago, June, *A general theory of linear sets.*

C. M. Jensen, Minnesota, June, physics, *Some problems in the approximate representation of a function by a Sturm-Liouville interpolation formula.*

C. G. Latimer, Chicago, August, *Arithmetic of generalized quaternions.*

Harry Levy, Princeton, June, *Tensors determined by a hypersurface in a Riemann space.*

J. B. Linker, Johns Hopkins, June, geophysics, *Equations of motion.*

L. H. McFarlan, Missouri, June, physics, *Transformation of the Euler equations in the calculus of variations.*

Aristotle Michal, Rice Institute, June, (a) *Integro-differential expressions invariant under Volterra group of transformations*; (b) *Functions of curves invariant under point transformations of the plane.*

J. A. Nyswander, Chicago, June, *A direct method of obtaining the solutions of systems of linear differential equations having constant coefficients.*

R. G. Putnam, Chicago, June, *On solutions of special, linear, third-order differential systems.*

B. P. Reinsch, Illinois, June, mechanics, *Expansion problems in connection with the hypergeometric differential equation.*

D. S. Saund, California, May, analytic mechanics, *The periodic solutions of the equation of the elliptic cylinder.*

J. H. Taylor, Chicago, June, *A generalisation of Levi-Civita's parallelism and the Frenet formulas.*

Marian M. Torrey, Cornell, June, physics, *On monoidal space transformations in which the monoids have a fixed tangent cone.*

D. V. Widder, Harvard, *Theorems of mean value and trigonometric interpolation.*

The following degree, conferred in 1923, should have been included in our list for that year (this BULLETIN, vol. 30, pp. 474–75):

Frederick Wood, Wisconsin, June, *Group-velocity and the propagation of disturbances in dispersive media.*

Professor E. A. Bailey, of La Grange College, has been appointed dean.

Dr. Henry Blumberg, of the University of Illinois, has been appointed professor of mathematics at Ohio State University.

At Brown University, Assistant Professor R. W. Burgess has resigned and will continue the statistical work with the Western Electric Company which he has been doing during the year 1924–25 while on leave of absence. Dr. C. R. Adams has been promoted to an assistant professorship.

At Cornell University, Assistant Professors W. B. Carver and D. C. Gillespie have been promoted to full professorships.

Assistant Professor C. A. Garabedian, of Northwestern University, has been appointed associate professor of mathematics in the University of Cincinnati.

At Harvard University, Professor J. L. Walsh has been granted leave of absence for the academic year 1925–26, and Professor G. D. Birkhoff for the second semester. Professor E. T. Bell has been appointed visiting lecturer for the first half-year.

Professor J. N. Michie, of the University of Texas, has been appointed head of the department of mathematics at Texas Technology College.

Associate Professor W. L. Miser, of the Armour Institute of Technology, has been appointed professor of mathematics at Vanderbilt University.

Professor D. E. Smith, of the faculty of pure science and Teachers College, Columbia University, retires at the end of the first semester of 1925–26. Dr. W. D. Reeve has been appointed associate professor of mathematics at Teachers College.

Professor F. B. Wiley, of Denison University, served as acting head of the department of mathematics at Roberts College, Constantinople, for the academic year 1924–25.

Assistant Professor L. T. Wilson, of the United States Naval Academy, has been promoted to an associate professorship.

Professor Alexander Ziwet, of the University of Michigan, has retired from active teaching.

The following appointments to instructorships are announced:

Cornell University, Messrs. R. P. Agnew, H. I. Lane, J. Wolfender, and Miss F. Farnum.

Harvard University, Dr. H. W. Brinkmann, Mr. H. L. Garabedian, Dr. L. M. Graves (first semester), Mr. B. O. Koopman (Peirce instructor), Mr. H. B. Hammatt, Mr. W. A. Jenkins, Mr. Malcolm MacClaren, Mr. Morris Marden, Mr. F. W. Perkins, Mr. H. P. Stabler, Mr. D. E. Whitford.

The death is announced of Professor C. Guichard, of the University of Paris.

Professor P. A. Nekrassoff, of the University of Moscow, died December 20, 1924, at the age of seventy-three.

Mr. W. W. Rouse Ball, fellow and formerly tutor of Trinity College, Cambridge, died April 4, 1925, at the age of seventy-four.

Dr. Oliver Heaviside, known for his work in electromagnetism, died February 3, 1925, in his seventy-fifth year.

Professor P. A. Lambert, of Lehigh University, died February 15, 1925, at the age of sixty-three. Professor Lambert had been a member of the American Mathematical Society since 1893.

Dr. D. J. McAdam, professor emeritus of mathematics at Washington and Jefferson College, died February 15, 1925, at the age of eighty-two.

Shirely E. Roberts, formerly professor of mathematics at the University of the Philippines, died March 9, 1925, at the age of forty-seven.

Professor C. S. Sperry, of the University of Colorado, died in July, 1924. Professor Sperry had been a member of the American Mathematical Society since 1921.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS.

BERNOULLI (J.). Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691/92. Nach der in der Basler Universitätsbibliothek befindlichen Handschrift übersetzt, mit einem Vorwort und Anmerkungen versehen von P. Schafheitlin. (Ostwalds Klassiker der exakten Wissenschaften, Nr. 211.) Leipzig, Akademische Verlagsgesellschaft, 1924. 56 pp.

BOREL (E.). Principes d'algèbre et d'analyse. Paris, Albin Michel, 1924. 8+310 pp.

GRELLING (K.). Mengenlehre. (Mathematisch-physikalische Bibliothek, Nr. 58.) Leipzig, Teubner, 1924. 49 pp.

GRISAR (C. M.). Über eine Verallgemeinerung des Tauberschen Satzes und seine Ausdehnung auf n-fache Reihen. (Diss.) München, 1924.

INSOLERA (F.). Complementi di matematiche generali. Torino, Lattes, 1924. 12+256 pp.

NIKOLAJEW (B.). Bedingungen der Möglichkeit der Gleichungen $x^n+y^n=z^n$ in ganzen Zahlen. Leningrad, 1924.

PASCH (M.). Mathematik und Logik. Vier Abhandlungen. 2te Auflage. Leipzig, Engelmann, 1924. 46 pp.

PETERS (L.). Vektoranalysis. (Mathematisch-physikalische Bibliothek, Nr. 57.) Leipzig, Teubner, 1924.

PÓLYA (G.) und SZEGÖ (G.). Aufgaben und Lehrsätze aus der Analysis. Band 1: Reihen, Integralrechnung, Funktionentheorie. (Die Grundlehren der mathematischen Wissenschaften, Band 19.) Berlin, Springer, 1925. 16+338 pp.

SCHAFHEITLIN (J.). See BERNOULLI (J.).

SCHWATT (I. J.). An introduction to the operations with series. Philadelphia, University of Pennsylvania Press, 1924. 10+287 pp.

SCHWERDT (H.). Lehrbuch der Nomographie. Berlin, Springer, 1924. 8+267 pp.

SZEGÖ (G.). See PÓLYA (G.).

THOMAS (R. G.). Essentials of applied calculus. New York, Van Nostrand, 1924.

THOMPSON (A. J.). Logarithmetica Britannica; being a standard table of logarithms to twenty decimal places. Part 9 (the first to be published): Numbers 90,000 to 100,000. London, Cambridge University Press, 1924. 4to. 18+100 pp.

WIELEITNER (H.). Die Geburt der modernen Mathematik. I: Analytische Geometrie. Karlsruhe, Braun, 1924 (?). 61 pp.

PART II. APPLIED MATHEMATICS

ANNUAIRE pour l'an 1925 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1925. 16mo. 686 pp.

AUERBACH (F.). Die Methoden der theoretischen Physik. Leipzig, Akademische Verlagsgesellschaft, 1925. 10+436 pp.

BACH (C.) und BAUMANN (R.). Elastizität und Festigkeit. 9te vermehrte Auflage. Berlin, Springer, 1924. 687 pp.

BARKHAUSEN (H.). Elektronenröhren. Band 1: Elektronentheoretische Grundlagen. Verstärker. 2te umgearbeitete Auflage. Leipzig, Hirzel, 1924. 6+140 pp.

BARRIOL (A.). Théorie et pratique des opérations financières. 3e édition. Paris, Doin, 1925. 400 pp.

BAUMANN (R.). See BACH (C.).

BENEDICKS (C.). Space and time. An experimental physicist's conception of these ideas and of their alteration. London, Methuen, 1924. 14+98 pp.

BERGET (A.). See CHAPPUIS (J.).

BINGEN (J. N.) et CROMBEZ (R.). Unités de mesure scientifiques et industrielles. Bruxelles, 1923. 47 pp.

BLEICH (F.). Theorie und Berechnung der eisernen Brücken. Berlin, Springer, 1924. 593 pp.

BOEHM (F.). Versicherungs-Mathematik. I: Elemente der Versicherungsrechnung. (Sammlung Göschen.) Berlin, de Gruyter, 1925. 144 pp.

BOQUET (F.). Histoire de l'astronomie. Paris, Payot, 1925. 509 pp.

BORN (M.). Vorlesungen über Atommechanik. Herausgegeben unter Mitwirkung von F. Hund. Band 1. (Struktur der Materie in Einzeldarstellungen, Band 2.) Berlin, Springer, 1925. 367 pp.

BOUASSE (H.). Houle, rides, seiches et marées. Paris, Delagrave, 1924. 540 pp.

——. Oscillations électriques. Paris, Delagrave, 1925. 375 pp.

BREISIG (F.). Theoretische Telegraphie. Eine Anwendung der Maxwellschen Elektrodynamique auf Vorgänge in Leitungen und Schaltungen. Braunschweig, Vieweg, 1924. 14+548 pp.

BRUCKNER (A.). Grundzüge der Brillenlehre. Band 1. Berlin, Springer, 1924. 167 pp.

CABREIRA (A.). Méthode pour obtenir les coordonnées géographiques et astrobales mécanique et solaire. Lisbonne, 1924. 18 pp.

CHAPPUIS (J.) et BERGET (A.). Leçons de physique générale. 2e édition entièrement refondue. Tome 1 et tome 3. Paris, Gauthier-Villars, 1924. 12+692+7+502 pp.

CLERC (A.). See JEANS (J. H.).

CROMBEZ (R.). See BINGEN (J. N.).

DANNEMANN (F.). Die Naturwissenschaften in ihrer Entwicklung und in ihrem Zusammenhange. 2te Auflage. Band IV: Das Emporblühen der modernen Naturwissenschaften seit der Entdeckung des Energieprinzips. Leipzig, Engelmann, 1923. 12 + 630 pp.

DEBYE (P.). See MARX (E.).

DRIESCH (H.). Relativitätstheorie und Philosophie. Karlsruhe, G. Braun, 1924. 52 pp.

EDDINGTON (A. S.). Vues générales sur la théorie de la relativité. Traduction par T. Greenwood. Préface de P. Painlevé. Paris, Gauthier-Villars, 1924. 24 + 103 pp.

EINSTEIN (A.). Quatre conférences sur la théorie de la relativité. Traduction de M. Solovine. Paris, Gauthier-Villars, 1925. 96 pp.

—— Sur l'électrodynamique des corps en mouvement. Traduction de M. Solovine. Paris, Gauthier-Villars, 1925. 56 pp.

FORCHHEIMER (P.). Hydraulik. 2te Auflage. Leipzig, Teubner, 1924. 10 + 566 pp.

GREENWOOD (T.). See EDDINGTON (A. S.).

GRIFFITH (R. O.). See LEWIS (W. C. M.).

HAAS (A.). Einführung in die theoretische Physik. 3te und 4te völlig umgearbeitete Auflage. Band 2. Berlin, de Gruyter, 1924. 379 pp.

VON HELMHOLTZ (H.). Treatise on physiological optics. Translated from the third German edition by J. P. C. Southall. Volumes 1-2. Ithaca, Optical Society of America, 1924. 9 + 480 pp.

HERZ (N.). Allgemeine Theorie zentrierter Linsensysteme. Leipzig und Wien, Deuticke, 1924. 4 + 68 pp.

HUND (F.). See BORN (M.).

HUNTER (H.). The chemical significance of optical dispersion. (Diss., University of London.) London, the Author, 1924. 117 pp.

INTERNATIONAAL CONGRES voor Technische Mechanica, Delft, 22.–28. April 1924. Uittreksels der Voodrachten. 131 pp.

JACQUEMIN (R.). See NICEFORO (A.).

JEANS (J. H.). Théorie dynamique des gaz. Traduit sur la 3e édition anglaise par A. Clerc. Paris, Blanchard, 1925. 510 pp.

JOOS (G.). See MARX (E.).

KLEINERT (H.). Die Prüfungsmöglichkeiten der Einsteinschen Relativitätstheorie. Allgemein verständliche Darstellung. Bern und Leipzig, Bircher, 1924. 63 pp.

LAMB (H.). Statics including hydrostatics and the elements of the theory of elasticity. 2d edition. Cambridge, University Press, 1924.

LANG (E.). See SPILLMAN (W. J.).

VON LAUE (M.). See MARX (E.).

LECHER (E.). See RIECKE (E.).

LEREDU (R.). L'équivoque d'Einstein. Paris, Les Presses Universitaires de France, 1925. 66 pp.

LORENTZ (H. A.). See MARX (E.).

LEWIS (W. C. M.). A system of physical chemistry. Volume 3: Quantum theory. With certain appendices by James Rice, A. M'Keown, and R. O. Griffith. London, Longmans, 1924. 10 + 407 pp.

LORENZ (H.). Lehrbuch der technischen Physik. Band 1, Teil 1: Mechanik ebener Gebilde. 2te, vollständig neubearbeitete Auflage der Technischen Mechanik starrer Systeme. Berlin, Springer, 1924. 288 pp.

LUDWIG (W.). Lehrbuch der darstellenden Geometrie. Teil 3: Das rechtwinklige Zweitafelsystem (Krumme, Flächen, Axonometrie, Perspektive). Berlin, Springer, 1924. 174 pp.

M'KEOWN (A.). See LEWIS (W. C. M.).

MANES (A.). Versicherungswesen. 4te neudurchgesehene Auflage. Band 1: Allgemeine Versicherungslehre. Band 2: Besondere Versicherungslehre. Leipzig, Teubner, 1924. 14 + 231 + 14 + 357 pp.

MARX (E.), herausgegeben von. Handbuch der Radiologie. Band 6: Die Theorien der Radiologie. Bearbeitet von M. von Laue, P. Zeeman, H. A. Lorentz, A. Sommerfeld, G. Wentzel, G. Joos, E. Riecke, L. Vegard, P. Debye. Leipzig, Akademische Verlagsgesellschaft, 1925. 11 + 806 pp.

MASCART (J.). Le plus vieil instrument européen d'astronomie nautique (IIIe siècle). Lyon, 1923. 16 pp.

MARTIN (L. C.). Optical measuring instruments: their construction, theory and use. London, Blackie, 1924. 9 + 270 pp.

MAURAIN (C.). Physique du globe. Paris, Colin, 1923. 204 pp.

MENZEL (H.). Die Theorie der Verbrennung, die stöchiometrischen und thermochemischen Grundlagen der Verbrennungs- und Vergasungs-Vorgänge. Dresden, Steinkopff, 1924. 8 + 120 pp.

MILLIKAN (R. A.). The electron. 2d edition. Chicago, University of Chicago Press, 1924. 14 + 293 pp.

MILLOT (S.). Théorie nouvelle de la probabilité des causes. Paris, Gauthier-Villars, 1925.

MILLS (F. C.). Statistical methods applied to economics and business. New York, Holt, 1924. 16 + 604 pp. $3.50

MOLESWORTH (F. C.). The military uses of astronomy. London, Longmans, 1924. 12 + 112 pp.

NICEFORO (A.). La méthode statistique et ses applications aux sciences naturelles, aux sciences sociales et à l'art. Traduit de l'italien par R. Jacquimin. Paris, Marcel Girard, 1925. 8 + 652 pp.

NOYES (A.). See WASHBURN (E. W.).

PAINLEVÉ (P.). See EDDINGTON (A. S.).

PETZOLDT (J.). Die Stellung der Relativitätstheorie in der geistigen Entwicklung der Menschheit. 2te verbesserte und vermehrte Auflage. Leipzig, Barth, 1923. 4 + 98 pp.

RICE (J.). See LEWIS (W. C. M.).

RIECKE (E.). Lehrbuch der Physik. 7te Auflage, herausgegeben von E. Lecher. Band 1. Berlin, de Gruyter, 1923. 16 + 656 pp.

—— See MARX (E.).

ROUSSILHE (H.). Cours d'astronomie appliquée et géodésie professé à l'Ecole spéciale des travaux publics, du bâtiment et de l'industrie. Paris, Léon Eyrolles, 1924.

SCHOUTEN (J. A.). Raum, Zeit und Relativitätsprinzip. Leipzig, Teubner, 1924. 40 pp.

SCHOY (K.). Die Gnomik der Araber. (Die Geschichte der Zeitmessung, Band 1.) Berlin, de Gruyter, 1923.

SILBERSTEIN (L.). The theory of relativity. 2d edition, enlarged. London, Macmillan, 1924. 10+563 pp.

SOLOVINE (M.). See EINSTEIN (A.).

SOMMERFELD (A.). See MARX (E.).

SOUTHALL (J. P. C.). See VON HELMHOLTZ (H.).

SPILLMAN (W. J.) and LANG (E.). The law of diminishing returns. Yonkers, World Book Company, 1924. 12+178 pp.

TAKABEYA (E.). Zur Berechnung des beiderseits eingemauerten Trägers unter besonderer Berücksichtigung der Längskraft. Berlin, Springer, 1924. 4+52 pp.

THOMSON (G. H.). How to calculate correlations. A non-mathematical book of instructions. London, Harrap, 1925. 24 pp.

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